6

The Fundamental Origin of Curvature and Torsion

by

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Abstract

The fundamental origin of curvature and torsion is discussed in terms of commutators of covariant derivatives in the general manifold. Detailed proofs are given of the origin of the curvature and torsion tensors. A proof of the Jacobi operator identity is given and this identity is used to prove that the conventional second Bianchi identity is true if and only if torsion is zero, and if and only if accompanied by a novel operator identity neglected in the literature. Finally group theoretical considerations are used to prove that in the case of rotation, the Riemann and torsion tensors can be interpreted as group structure constants. This proof in turn leads to the conclusion that the equations of classical electrodynamics take the same vectorial form in Einstein Cartan Evans (ECE) and Maxwell Heaviside field theory, but in different base manifolds.

Keywords: Curvature, torsion, covariant derivatives, Jacobi identity, Bianchi identity, rotational origin of Cartan torsion, Einstein Cartan Evans (ECE) field theory.

6.1 Introduction

Recently a generally covariant unified field theory has been developed [1–8] in which the electromagnetic sector is represented by Cartan geometry in which appears curvature and torsion. This theory is known as Einstein Cartan

Evans (ECE) theory because it is based on an extension of the Riemann geometry used by Einstein to include Cartan's torsion [9]. In ECE theory the electromagnetic field is directly proportional to the Cartan torsion. The latter is represented in Cartan differential geometry by a vector valued differential two-form defined by:

$$T^a = d \wedge q^a + \omega^a{}_b \wedge q^b \tag{6.1}$$

where q^a is the Cartan tetrad, a vector valued differential one-form [9], and where $\omega^a{}_b$ is the spin connection. The index a is defined in a tangential Minkowski space-time at a point P in the base manifold, a manifold which represents a four dimensional space-time with torsion and curvature. Using the tetrad postulate [1–9]:

$$D_\mu q^a_\nu = 0 \tag{6.2}$$

the definition (6.1) becomes equivalent to the definition of the Cartan torsion tensor in the base manifold:

$$T^\kappa_{\mu\nu} = \Gamma^\kappa_{\mu\nu} - \Gamma^\kappa_{\nu\mu} \tag{6.3}$$

where $\Gamma^\kappa_{\mu\nu}$ is the connection of the base manifold. In the Riemann geometry used by Einstein to develop general relativity the connection is the Christoffel connection:

$$\Gamma^\kappa_{\mu\nu} = \Gamma^\kappa_{\nu\mu} \tag{6.4}$$

so in Einsteinian general relativity and cosmology, torsion is zero. In ECE theory the electromagnetic potential and field are directly proportional respectively to the tetrad and torsion forms:

$$A^a_\mu = A^{(0)} q^a_\mu \tag{6.5}$$
$$F^a_{\mu\nu} = A^{(0)} T^a_{\mu\nu}. \tag{6.6}$$

Therefore in the base manifold, the electromagnetic field becomes a rank three tensor proportional to the torsion tensor:

$$F^\kappa_{\mu\nu} = A^{(0)} T^\kappa_{\mu\nu}. \tag{6.7}$$

It has been shown [1–8] that the definition (6.7) leads to the equations of classical electrodynamics in the same vectorial notation as the Maxwell Heaviside vector theory but written in a base manifold with torsion and curvature, not in the Minkowski space-time. The Cartesian components of the electric

and magnetic field in ECE theory are defined by elements of the rank three torsion tensor as follows:

$$\left.\begin{array}{ll} E_X = E^1{}_{01}, & B_X = B^1{}_{23}, \\ E_Y = E^2{}_{02}, & B_Y = B^2{}_{31}, \\ E_Z = E^3{}_{03}, & B_Z = B^3{}_{12}. \end{array}\right\} \tag{6.8}$$

In this paper a further fundamental proof of equation (6.8) is given using the fundamental definition [9] of the Riemann tensor and torsion tensor in terms of commutators of covariant derivatives (or round trip in the base manifold). The derivation of the Riemann tensor and torsion tensor (6.3) using this method is given in detail in Section 6.2. In Section 6.3 the proof of the Jacobi identity is given. The Jacobi identity [9–10] is exact and is valid both for covariant derivatives and group generators [10]. In this section the Jacobi identity is used to show under what circumstances the second Bianchi identity [9] is valid. This is important because the second Bianchi identity is used directly in the derivation of the Einstein Hilbert (EH) field equation. It is found that the second Bianchi identity and EH field equation are valid if and only if the Cartan torsion tensor (6.3) is zero, and if and only if:

$$\begin{gathered} R^\rho{}_{\sigma\mu\nu}D_\kappa + R^\rho{}_{\sigma\kappa\mu}D_\nu + R^\rho{}_{\sigma\nu\kappa}D_\mu = 0, \\ R^a{}_b \wedge D = 0, \end{gathered} \tag{6.9}$$

which is a new differential operator relation which appears to have been hitherto neglected in the literature. In Eq. (6.9), $R^a{}_b$ is the curvature or Riemann differential form [1–9] and D represents the covariant derivative in differential geometry. The conventional second Bianchi identity is usually written as the reverse of Eq. (6.9):

$$D \wedge R^a{}_b = 0. \tag{6.10}$$

In the presence of the torsion form however the rigorously correct Bianchi identity is [1–9]:

$$D \wedge T^a := R^a{}_b \wedge q^b \tag{6.11}$$

and there is only one Bianchi identity (see paper 88 of www.aias.us). The second one can be derived from Eq. (6.11).

In Section 6.4 finally, the rotational limit of Cartan geometry is considered using the round trip method and it is shown that in this limit the Cartan torsion tensor (6.3) can be considered to be a group structure constant. These considerations lead directly to the interpretation (6.8) of the electric and magnetic field components in ECE theory (see papers 93 and following on www.aias.us).

6.2 Derivation of the Curvature and Torsion Tensors from Commutators of Covariant Derivatives

Although this proof is well known it is usually given in textbooks [9] without sufficient detail for understanding by non-specialists. It gives a fundamental interpretation for both curvature and torsion in terms of a round trip in the general base manifold. This method is also used in field theory [10] and so the curvature, torsion and field tensors have the same fundamental origin. This is therefore a fundamental justification for the basic ECE hypothesis, that the electromagnetic field tensor is directly proportional to the Cartan torsion, they are both commutators of covariant derivatives. The origin of both the Riemann and Cartan torsion tensors is parallel transport around a closed loop:

$$\delta V^\rho = (\delta a)(\delta b) A^\nu B^\mu R^\rho{}_{\sigma\mu\nu} V^\sigma \tag{6.12}$$

which can be represented by a commutator of covariant derivatives [9–10]. The covariant derivative of a tensor in a given direction measures [9] how much the tensor changes relative to what it would have been if it had been parallel transported. The commutator of covariant derivatives measures the difference between parallel transporting the tensor one way and then the other, versus the opposite ordering. In flat or Minkowski space-time the result is zero, it makes no difference which sense the process takes place. In flat space-time the covariant derivatives become ordinary derivatives so the following operator is zero:

$$[\partial_\mu, \partial_\nu] = \partial_\mu \partial_\nu - \partial_\nu \partial_\mu = 0. \tag{6.13}$$

The commutator of covariant derivatives is however an operator which is not zero:

$$[D_\mu, D_\nu] \neq 0. \tag{6.14}$$

Such commutators are well known in the theory of rotation generators, angular momentum, group theory and quantum mechanics [10]. They also appear in differential geometry as the wedge product [1–9] of two one-forms, which is defined by:

$$A^a \wedge B^b = [A^a_\mu, A^b_\nu]. \tag{6.15}$$

The Riemann and torsion tensors are defined [9] by:

$$[D_\mu, D_\nu] V^\rho = D_\mu (D_\nu V^\rho) - D_\nu (D_\mu V^\rho) \tag{6.16}$$

where V^ρ is a four vector in a base manifold with curvature and torsion. On the right hand side of Eq. (6.16) the covariant derivatives act on rank two tensors contained within the brackets. The rule [1–9] for the covariant derivative of a rank two tensor then gives:

$$[D_\mu, D_\nu]V^\rho = \partial_\mu(D_\nu V^\rho) - \Gamma^\lambda_{\mu\nu}D_\lambda V^\rho + \Gamma^\rho_{\mu\sigma}D_\nu V^\sigma$$
$$- \partial_\nu(D_\mu V^\rho) + \Gamma^\lambda_{\nu\mu}D_\lambda V^\rho - \Gamma^\rho_{\nu\sigma}D_\mu V^\sigma \qquad (6.17)$$

within which are defined:

$$\left.\begin{aligned} D_\nu V^\rho &= \partial_\nu V^\rho + \Gamma^\rho_{\ \nu\lambda}V^\lambda, \\ D_\lambda V^\rho &= \partial_\lambda V^\rho + \Gamma^\rho_{\ \lambda\sigma}V^\sigma, \\ D_\nu V^\sigma &= \partial_\nu V^\sigma + \Gamma^\sigma_{\ \nu\lambda}V^\lambda. \end{aligned}\right\} \qquad (6.18)$$

Therefore there are equations such as:

$$\partial_\mu(D_\nu V^\rho) = \partial_\mu\partial_\nu V^\rho + (\partial_\mu\Gamma^\rho_{\ \nu\lambda})V^\lambda + \Gamma^\rho_{\ \nu\lambda}\partial_\mu V^\lambda$$
$$= \partial_\mu\partial_\nu V^\rho + (\partial_\mu\Gamma^\rho_{\ \nu\sigma})V^\sigma + \Gamma^\rho_{\ \nu\sigma}\partial_\mu V^\sigma. \qquad (6.19)$$

The dummy or summation indices are now re-arranged as follows:

$$\lambda \to \sigma \qquad (6.20)$$

This gives:

$$[D_\mu, D_\nu]V^\rho = \partial_\mu\partial_\nu V^\rho + (\partial_\mu\Gamma^\rho_{\ \nu\sigma})V^\sigma + \Gamma^\rho_{\ \nu\sigma}\partial_\mu V^\sigma$$
$$- \Gamma^\lambda_{\mu\nu}\partial_\lambda V^\rho - \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\ \lambda\sigma}V^\sigma$$
$$+ \Gamma^\rho_{\mu\sigma}\partial_\nu V^\sigma + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\ \nu\lambda}V^\lambda$$
$$- \partial_\nu\partial_\mu V^\rho - (\partial_\nu\Gamma^\rho_{\ \mu\sigma})V^\sigma - \Gamma^\rho_{\mu\sigma}\partial_\nu V^\sigma$$
$$+ \Gamma^\lambda_{\nu\mu}\partial_\lambda V^\rho + \Gamma^\lambda_{\nu\mu}\Gamma^\rho_{\ \lambda\sigma}V^\sigma$$
$$- \Gamma^\rho_{\mu\sigma}\partial_\nu V^\sigma - \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\ \nu\lambda}V^\lambda \qquad (6.21)$$

which can be re-arranged to give:

$$[D_\mu, D_\nu]V^\rho = (\partial_\mu\Gamma^\rho_{\ \nu\sigma} - \partial_\nu\Gamma^\rho_{\ \mu\sigma} + \Gamma^\rho_{\ \mu\lambda}\Gamma^\lambda_{\ \nu\sigma} - \Gamma^\rho_{\ \nu\lambda}\Gamma^\lambda_{\ \mu\sigma})V^\sigma$$
$$- (\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu})(\partial_\lambda V^\rho + \Gamma^\rho_{\ \lambda\sigma}V^\sigma). \qquad (6.22)$$

Finally this is expressed as:

$$[D_\mu, D_\nu]V^\rho = R^\rho_{\ \sigma\mu\nu}V^\sigma - T^\lambda_{\mu\nu}D_\lambda V^\rho \qquad (6.23)$$

where the Riemann tensor is defined [9] by:

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \qquad (6.24)$$

and the torsion tensor is defined by:

$$T^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}. \qquad (6.25)$$

The overall result:

$$[D_{\mu}, D_{\nu}]V^{\rho} = R^{\rho}{}_{\sigma\mu\nu}V^{\sigma} - T^{\lambda}_{\mu\nu}D_{\lambda}V^{\rho} \qquad (6.26)$$

is true irrespective of the symmetry of the metric and connection, and irrespective of the metric compatibility condition [9]. The use of the commutator of covariant derivatives means the Riemann and torsion tensors are always anti-symmetric in their last two indices:

$$R^{\rho}{}_{\sigma\mu\nu} = -R^{\rho}{}_{\sigma\nu\mu}, \; T^{\lambda}_{\mu\nu} = -T^{\lambda}_{\nu\mu} \qquad (6.27)$$

indicating their rotational or commutative or anti-symmetric origin. Both curvature and torsion are kinds of rotation, or bending and twisting. However, it is important to note that there is no symmetry restriction on the first two indices of the Riemann tensor in general. The Riemann tensor is anti-symmetric in its first two indices if and only if the metric compatibility condition is used [9]. If it is assumed that the metric is symmetric:

$$g_{\mu\nu} = g_{\nu\mu}. \qquad (6.28)$$

It follows form metric compatibility [9] that the connection is symmetric:

$$\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu} \qquad (6.29)$$

and that torsion vanishes. In Cartan geometry [1–9], the torsion is not zero in general, so the metric and connection are not symmetric in general. The conventional first Bianchi identity is true if and only if the metric and connection are symmetric, and if and only if the torsion is zero. In differential form notation the first Bianchi identity in the absence of torsion is:

$$R^{a}{}_{b} \wedge q^{b} = 0 \qquad (6.30)$$

and in tensor notation it is:

$$R_{\sigma\mu\nu\rho} + R_{\sigma\rho\mu\nu} + R_{\sigma\nu\rho\mu} = 0. \qquad (6.31)$$

However it is important to note that Eqs. (6.30) and (6.31) are special cases. The rigorously correct first Bianchi identity is [1–9]:

$$D \wedge T^a := R^a{}_b \wedge q^b \tag{6.32}$$

not zero in general, and the rigorously correct second Bianchi identity is a re-expression of Eq. (6.32), and not an independent identity (paper 88 of www.aias.us). Historically, the first Bianchi identity was given by Ricci and Levi-Civita and not by Bianchi.

In the Minkowski space-time:

$$[D_\mu, D_\nu] = [\partial_\mu, \partial_\nu] = 0, \tag{6.33}$$

$$R^\rho{}_{\sigma\mu\nu} = T^\lambda_{\mu\nu} = 0, \tag{6.34}$$

and this is the space-time of Maxwell Heaviside field theory.

The Riemann and torsion tensors are constructed from the connection and are true for any connection, whether metric compatible or not. Although the connection is non-tensorial, the Riemann and torsion tensors are true tensors by construction [1–9] and all the equations of Riemann and Cartan geometry are generally covariant, i.e. tensorial under the general coordinate transformation. This means that equations of physics based on these geometries, such as ECE theory [1–8] are rigorously objective equations of physics, they are the same in form to an observer moving arbitrarily with respect to another, in a frame of reference moving arbitrarily with respect to another frame of reference. This is the essence of the essentially geometrical philosophy of general relativity as is well known [9]. The essence of the matter is that physics is geometry, all physics is geometry, not just gravitation. Otherwise we do not have a self consistent basic philosophy of physics. Maxwell Heaviside (MH) field theory does not obey this philosophy because the MH field theory is defined in Minkowski space-time, and MH theory is Lorentz covariant by construction. It contains no connection and is not generally covariant.

For a tensor of any rank [9]:

$$[D_\rho, D_\sigma]\chi^{\mu_1...\mu_k}_{\nu_1...\nu_l} = -T^\lambda_{\rho\sigma} D_\lambda \chi^{\mu_1...\mu_k}_{\nu_1...\nu_l} + R^{\mu_1}{}_{\lambda\rho\sigma}\chi^{\lambda\mu_2...\mu_k}_{\nu_1...\nu_l} + R^{\mu_2}{}_{\lambda\rho\sigma}\chi^{\mu_1\lambda...\mu_k}_{\nu_1...\nu_l} + \dots$$
$$- R^\lambda{}_{\nu_1\rho\sigma}\chi^{\mu_1...\mu_k}_{\lambda\nu_2...\nu_l} + R^\lambda{}_{\nu_2\rho\sigma}\chi^{\mu_1...\mu_k}_{\nu_1\lambda...\nu_l} - \dots \tag{6.35}$$

The commutator of two vector fields X and Y is a third vector field [9] with components:

$$[X, Y]^\mu = X^\lambda \partial_\lambda Y^\mu - Y^\lambda \partial_\lambda X^\mu. \tag{6.36}$$

The curvature and torsion tensors can be though of as multi-linear maps [9], the torsion being a map from two vector fields to a third:

$$T(X,Y) = D_X Y - D_Y X - [X,Y] \tag{6.37}$$

and the curvature as a map from three vector fields to a fourth [9]:

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z \tag{6.38}$$

where:

$$D_X = X^\mu D_\mu. \tag{6.39}$$

Cartan's geometry [1–9] expresses these results in an elegant and concise way through his two well known structure equations [9]:

$$T^a = D \wedge q^a, \tag{6.40}$$
$$R^a{}_b = D \wedge \omega^a{}_b. \tag{6.41}$$

6.3 The Jacobi and Bianchi Identities

The Jacobi identity [1–10] is an exact identity used in field theory and general relativity. It is an operator identity that applies to covariant derivatives [9] and group generators [10] alike. It is very rarely proven in all detail however and so the following is a detailed proof. It is necessary to prove that:

$$[[D_\lambda, D_\rho], D_\sigma] + [[D_\rho, D_\sigma], D_\lambda] + [[D_\sigma, D_\lambda], D_\rho] := 0 \tag{6.42}$$

which is the Jacobi identity, an exact identity. The proof expands the commutators as follows

$$
\begin{aligned}
\text{L.H.S} = {} & (D_\lambda D_\rho - D_\rho D_\lambda)D_\sigma - D_\sigma(D_\lambda D_\rho - D_\rho D_\lambda) \\
& + (D_\rho D_\sigma - D_\sigma D_\rho)D_\lambda - D_\lambda(D_\rho D_\sigma - D_\sigma D_\rho) \\
& + (D_\sigma D_\lambda - D_\lambda D_\sigma)D_\rho - D_\rho(D_\sigma D_\lambda - D_\lambda D_\sigma)
\end{aligned}
\tag{6.43}
$$

and this expansion is regarded as an expansion of algebra:

$$
\begin{aligned}
\text{L.H.S} = {} & D_\lambda D_\rho D_\sigma - D_\rho D_\lambda D_\sigma - D_\sigma D_\lambda D_\rho + D_\sigma D_\rho D_\lambda \\
& + D_\rho D_\sigma D_\lambda - D_\sigma D_\rho D_\lambda - D_\lambda D_\rho D_\sigma + D_\lambda D_\sigma D_\rho \\
& + D_\sigma D_\lambda D_\rho - D_\lambda D_\sigma D_\rho - D_\rho D_\sigma D_\lambda + D_\rho D_\lambda D_\sigma := 0
\end{aligned}
\tag{6.44}
$$

Q.E.D.

In field theory [10] the Jacobi identity is used to define field equations, and the commutator of covariant derivatives defines the field $G_{\mu\nu}$ through a constant g:

$$[D_\mu, D_\nu] = -igG_{\mu\nu}. \tag{6.45}$$

The idea of covariant derivative in field theory is borrowed from general relativity [10] and in condensed notation in field theory there exist commutators such as:

$$\begin{aligned}[D_\mu, D_\nu] &= [\partial_\mu - igA_\mu, \partial_\nu - igA_\nu] \\ &= -ig(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu])\end{aligned} \tag{6.46}$$

which have been extensively developed in to precursor theories of ECE such as O(3) electrodynamics (see Omnia Opera section of www.aias.us from 1992 to 2003). In Ryder's [10] eq. (3.173) for example there appears a field equation:

$$D_\rho G_{\mu\nu} + D_\mu G_{\nu\rho} + D_\nu G_{\rho\mu} = 0 \tag{6.47}$$

which in the notation of differential geometry [1–9] is:

$$D \wedge G = 0. \tag{6.48}$$

Eq. (6.48) is similar to the Bianchi identity of differential geometry, which becomes the ECE homogeneous field equation [1–8]:

$$D \wedge F^a = A^{(0)}(R^a{}_b \wedge q^b) \tag{6.49}$$

or

$$d \wedge F^a = A^{(0)}(R^a{}_b \wedge q^b - \omega^a{}_b \wedge T^b). \tag{6.50}$$

It is clear that both Ryder's field equation (6.48) and the ECE field equation (6.49) share a common origin in the commutator of covariant derivatives, but ECE theory is developed in a more general manifold than the type of field theory used by Ryder [10]. The latter is restricted to the Minkowski manifold only.

Restricting consideration of Eq. (6.26) to the torsion free case it becomes:

$$[D_\mu, D_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma \tag{6.51}$$

which can be expanded as:

$$[D_\mu, D_\nu]V^0 = R^0{}_{0\mu\nu}V^0 + R^0{}_{1\mu\nu}V^1 + R^0{}_{2\mu\nu}V^2 + R^0{}_{3\mu\nu}V^3$$
$$[D_\mu, D_\nu]V^1 = R^1{}_{0\mu\nu}V^0 + R^1{}_{1\mu\nu}V^1 + R^1{}_{2\mu\nu}V^2 + R^1{}_{3\mu\nu}V^3$$
$$[D_\mu, D_\nu]V^2 = R^2{}_{0\mu\nu}V^0 + R^2{}_{1\mu\nu}V^1 + R^2{}_{2\mu\nu}V^2 + R^2{}_{3\mu\nu}V^3$$
$$[D_\mu, D_\nu]V^3 = R^3{}_{0\mu\nu}V^0 + R^3{}_{1\mu\nu}V^1 + R^3{}_{2\mu\nu}V^2 + R^3{}_{3\mu\nu}V^3. \qquad (6.52)$$

In the torsion free case the following Riemann tensor elements vanish:

$$R^0{}_{0\mu\nu} = R^1{}_{1\mu\nu} = R^2{}_{2\mu\nu} = R^3{}_{3\mu\nu} = 0 \qquad (6.53)$$

because [9] in this case:

$$R^\rho{}_{\sigma\mu\nu} = -R^\sigma{}_{\rho\mu\nu}. \qquad (6.54)$$

Therefore

$$[D_\mu, D_\nu] \begin{bmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{bmatrix} = \begin{bmatrix} 0 & R^0{}_1 & R^0{}_2 & R^0{}_3 \\ -R^0{}_1 & 0 & R^1{}_2 & R^1{}_3 \\ -R^0{}_2 & -R^1{}_2 & 0 & R^2{}_3 \\ -R^0{}_3 & -R^1{}_3 & -R^2{}_3 & 0 \end{bmatrix}_{\mu\nu} \begin{bmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{bmatrix} \qquad (6.55)$$

and it is possible to define an operator equation similar to Eq. (6.45) of field theory:

$$[D_\mu, D_\nu] = R_{\mu\nu} := \begin{bmatrix} 0 & R^0{}_{1\mu\nu} & R^0{}_{2\mu\nu} & R^0{}_{3\mu\nu} \\ -R^0{}_{1\mu\nu} & 0 & R^1{}_{2\mu\nu} & R^1{}_{3\mu\nu} \\ -R^0{}_{2\mu\nu} & -R^1{}_{2\mu\nu} & 0 & R^2{}_{3\mu\nu} \\ -R^0{}_{3\mu\nu} & -R^1{}_{3\mu\nu} & -R^2{}_{3\mu\nu} & 0 \end{bmatrix} \qquad (6.56)$$

illustrating the relation between field theory and general relativity.

The conventionally named second Bianchi identity [9] may be derived from Eq. (6.51) as follows:

$$\begin{aligned} ([D_\kappa, [D_\mu, D_\nu]])V^\rho &= (D_\kappa[D_\mu, D_\nu] - [D_\mu, D_\nu]D_\kappa)V^\rho \\ &= D_\kappa(D_\mu(D_\nu V^\rho) - D_\nu(D_\mu V^\rho)) - D_\mu D_\nu(D_\kappa V^\rho) \\ &\quad + D_\nu D_\mu(D_\kappa V^\rho) \\ &= D_\kappa([D_\mu, D_\nu]V^\rho) - [D_\mu, D_\nu](D_\kappa V^\rho) \\ &= (D_\kappa[D_\mu, D_\nu])V^\rho + [D_\mu, D_\nu]D_\kappa V^\rho - [D_\mu, D_\nu]D_\kappa V^\rho \end{aligned}$$
$$(6.57)$$

using the Leibnitz Theorem. Therefore:

$$(D_\kappa, [D_\mu, D_\nu])V^\rho = (D_\kappa [D_\mu, D_\nu])V^\rho. \tag{6.58}$$

Therefore the Jacobi identity in this case becomes:

$$(D_\kappa [D_\mu, D_\nu] + D_\nu [D_\kappa, D_\mu] + D_\mu [D_\nu, D_\kappa])V^\rho = 0 \tag{6.59}$$

i.e.:

$$D_\kappa (R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda{}_{\mu\nu} D_\lambda V^\rho) + D_\nu (R^\rho{}_{\sigma\kappa\mu} V^\sigma - T^\lambda{}_{\kappa\mu} D_\lambda V^\rho)$$
$$+ D_\mu (R^\rho{}_{\sigma\nu\kappa} V^\sigma - T^\lambda{}_{\nu\kappa} D_\lambda V^\rho) = 0. \tag{6.60}$$

The conventional second Bianchi identity is a special case of this equation when the torsion vanishes, so:

$$D_\kappa (R^\rho{}_{\sigma\mu\nu} V^\sigma) + D_\nu (R^\rho{}_{\sigma\kappa\mu} V^\sigma) + D_\mu (R^\rho{}_{\sigma\nu\kappa} V^\sigma) = 0 \tag{6.61}$$

Using the Leibnitz Theorem:

$$D_\kappa (R^\rho{}_{\sigma\mu\nu} V^\sigma) = (D_\kappa R^\rho{}_{\sigma\mu\nu})V^\sigma + R^\rho{}_{\sigma\mu\nu} D_\kappa V^\sigma \tag{6.62}$$

and it is seen that the second Bianchi identity:

$$D_\kappa R^\rho{}_{\sigma\mu\nu} + D_\nu R^\rho{}_{\sigma\kappa\mu} + D_\mu R^\rho{}_{\sigma\nu\kappa} = 0 \tag{6.63}$$

is true if and only if the following operator identity is also true:

$$R^\rho{}_{\sigma\mu\nu} D_\kappa + R^\rho{}_{\sigma\kappa\mu} D_\nu + R^\rho{}_{\sigma\nu\kappa} D_\mu = 0 \tag{6.64}$$

In differential form notation Eq. (6.64) is [1–9]:

$$R^a{}_b \wedge D = 0 \tag{6.65}$$

and Eq. (6.63) is:

$$D \wedge R^a{}_b = 0. \tag{6.66}$$

Therefore both Eqs. (6.65) and (6.4) severely restrict the validity of EH general relativity and cosmology.

6.4 Pure Rotational Limit

In previous work [1–8] the pure rotational limit of ECE theory has been considered to be defined by the duality of $R^a{}_b$ and T^d in the Minkowski tangent space-time:

$$R^a{}_b = -\frac{\kappa}{2}\epsilon^a{}_{bd}T^d. \tag{6.67}$$

In this section the pure rotational limit is considered to be the special case from Eq. (6.26) where:

$$R^\rho{}_{\sigma\mu\nu}V^\rho = -T^\lambda{}_{\mu\nu}D_\lambda V^\rho \tag{6.68}$$

which is similar to Eq. (6.67) but written in the base manifold. With Eq. (6.26), Eq. (6.68) becomes a rotation generator type equation:

$$[D_\mu, D_\nu]V^\rho = -2T^\lambda{}_{\mu\nu}D_\lambda V^\rho \tag{6.69}$$

giving the operator equation:

$$[D_\mu, D_\nu] = -2T^\lambda{}_{\mu\nu}D_\lambda \tag{6.70}$$

in which the covariant derivatives obey the Jacobi identity (6.42). The covariant derivatives appearing in Eq. (6.70) can also be considered as group generators [9–10]. For example the group generators of SO(3) obey:

$$[I_i, I_j] = i\epsilon_{ijk}I_k \tag{6.71}$$

where the group structure constant [10] is:

$$C_{ijk} = i\epsilon_{ijk} \tag{6.72}$$

and:

$$C_{lim}C_{mjk} + C_{ljm}C_{mki} + C_{lkm}C_{mij} = 0. \tag{6.73}$$

The group structure constant is defined by the adjoint representation [10]

$$C_{imn} = (I_i)_{mn} \tag{6.74}$$

where:

$$I_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, I_2 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, I_3 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (6.75)$$

There is a clear similarity between Eqs. (6.70) and (6.71), except that Eq. (6.70) is written in the general manifold and Eq. (6.71) is Euclidean. Therefore it is possible to think of the torsion tensor in Eq. (6.70) as a group structure constant in the general manifold. The generators of the group are the covariant derivatives in the general manifold. Taking the analogy further the SU(3) group [10] is defined by Gell-Mann matrices which obey the commutator relation:

$$\left[\frac{\lambda a}{2}, \frac{\lambda b}{2} \right] = i f_{abc} \frac{\lambda c}{2}. \quad (6.76)$$

The group structure constant in this case is defined by [10] if_{abc}.

It therefore follows that if Eq. (6.70) is considered to be rotational in nature, analogous to Eqs. (6.71) or (6.76), the possible values of $T^{\lambda}_{\mu\nu}$ are the totally anti-symmetric T^1_{23}, T^2_{31} and T^3_{12}. These are space-like and play a role analogous to ϵ_{123}, ϵ_{231}, and ϵ_{312} in Euclidean space-time. If the upper indices are held constant and Hodge duals are performed on the lower two indices we obtain T^1_{01}, T^2_{02}, and T^3_{03}. In ECE theory [1–8] they define the components of the electric and magnetic fields of the generally covariant electromagnetic sector (Eqs. (6.8) and (6.50)). In the case of pure rotation the electro-dynamical equations of ECE in vector notation are the free space values:

$$\left. \begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{B} &= 0 \\ \boldsymbol{\nabla} \times \boldsymbol{E} + \partial \boldsymbol{B}/\partial t &= \mathbf{0} \\ \boldsymbol{\nabla} \cdot \boldsymbol{E} &= 0 \\ \boldsymbol{\nabla} \times \boldsymbol{B} - \frac{1}{c^2} \partial \boldsymbol{E}/\partial t &= \mathbf{0}. \end{aligned} \right\} \quad (6.77)$$

More generally in the laboratory, and for all practical purposes, they become [1–8] the familiar vectorial laws:

$$\left. \begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{B} &= 0 \\ \boldsymbol{\nabla} \times \boldsymbol{E} + \partial \boldsymbol{B}/\partial t &= \mathbf{0} \\ \boldsymbol{\nabla} \cdot \boldsymbol{E} &= \rho/\epsilon_0 \\ \boldsymbol{\nabla} \times \boldsymbol{B} - \frac{1}{c^2} \partial \boldsymbol{E}/\partial t &= \mu_0 \mathbf{J} \end{aligned} \right\} \quad (6.78)$$

but now written in the general manifold.

Acknowledgments

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