# Appendices

## I. Verification of Equation (7)

$$\tilde{\tilde{T}}^{\rho}_{\ \mu\nu} = \tilde{T}_{\lambda} = T^{\rho}_{\mu\nu}$$

First we create a d4-vector, take its Hodge Dual then take the Hodge Dual of the result.

```
Πvector = {Π1, Π2, Π3, Π4};
Print["Πvector    =   ", Πvector // MatrixForm]
Πtensor = −HodgeDual[Πvector];
Print["Πtensor = −Hodge Dual(Πvector)   =  ", % // MatrixForm]
HodgeDual[%%];
Print["Hodge Dual(Πtensor) = ", % // MatrixForm]
```

Πvector $=\begin{pmatrix}\Pi1\\ \Pi2\\ \Pi3\\ \Pi4\end{pmatrix}$

Πtensor = −Hodge Dual(Πvector) $=\begin{pmatrix}\begin{pmatrix}0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\\Pi4\\-\Pi3\end{pmatrix} & \begin{pmatrix}0\\-\Pi4\\0\\\Pi2\end{pmatrix} & \begin{pmatrix}0\\\Pi3\\-\Pi2\\0\end{pmatrix}\\ \begin{pmatrix}0\\0\\-\Pi4\\\Pi3\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}\Pi4\\0\\0\\-\Pi1\end{pmatrix} & \begin{pmatrix}-\Pi3\\0\\\Pi1\\0\end{pmatrix}\\ \begin{pmatrix}0\\\Pi4\\0\\-\Pi2\end{pmatrix} & \begin{pmatrix}-\Pi4\\0\\0\\\Pi1\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}\Pi2\\-\Pi1\\0\\0\end{pmatrix}\\ \begin{pmatrix}0\\-\Pi3\\\Pi2\\0\end{pmatrix} & \begin{pmatrix}\Pi3\\0\\-\Pi1\\0\end{pmatrix} & \begin{pmatrix}-\Pi2\\\Pi1\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\end{pmatrix}\end{pmatrix}$

Hodge Dual(Πtensor) $=\begin{pmatrix}\Pi1\\ \Pi2\\ \Pi3\\ \Pi4\end{pmatrix}$

## II. Creating $\Gamma^{\rho}_{\mu\nu}$ as Sum of Totally Antisymmetric Tensor and Purely Diagonal Tensor

Create the diagonal array $\pi^{\rho}_{\mu\nu}$ then set off − diagonals to zero, and diagnals to match equation (17).

```
SymmetrizedArray[pos_ :→ Λ_pos, {4, 4, 4}, Symmetric[{1, 2, 3}]] -
    SymmetrizedArray[pos_ :→ Λ_pos, {4, 4, 4}, Antisymmetric[{1, 2, 3}]];
% /. Λ_{1,2,4} → 0;
% /. Λ_{1,2,3} → 0;
% /. Λ_{1,3,4} → 0;
% /. Λ_{2,3,4} → 0;
% /. Λ_{1,1,1} → Λ1;
% /. Λ_{1,2,2} → Λ1;
% /. Λ_{1,3,3} → Λ1;
% /. Λ_{1,4,4} → Λ1;

% /. Λ_{2,1,1} → Λ2;
% /. Λ_{2,2,2} → Λ2;
% /. Λ_{2,3,3} → Λ2;
% /. Λ_{2,4,4} → Λ2;

% /. Λ_{3,1,1} → Λ3;
% /. Λ_{3,2,2} → Λ3;
% /. Λ_{3,3,3} → Λ3;
% /. Λ_{3,4,4} → Λ3;

% /. Λ_{4,1,1} → Λ4;
% /. Λ_{4,2,2} → Λ4;
% /. Λ_{4,3,3} → Λ4;
% /. Λ_{4,4,4} → Λ4;

% /. Λ_{1,1,2} → Λ2;
% /. Λ_{1,1,3} → Λ3;
% /. Λ_{1,1,4} → Λ4;

% /. Λ_{2,2,3} → Λ3;
% /. Λ_{2,2,4} → Λ4;

% /. Λ_{3,3,4} → Λ4;

Λform = %;
Print["π^ρ_μν    =", Λform // MatrixForm]
TensorSymmetry[%%];
Print["π^ρ_μν symmetry is ", %]
```

$$\pi^{\rho}_{\mu\nu} \;\; = \begin{pmatrix} \begin{pmatrix} \Lambda1 \\ \Lambda2 \\ \Lambda3 \\ \Lambda4 \end{pmatrix} & \begin{pmatrix} \Lambda2 \\ \Lambda1 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \Lambda3 \\ 0 \\ \Lambda1 \\ 0 \end{pmatrix} & \begin{pmatrix} \Lambda4 \\ 0 \\ 0 \\ \Lambda1 \end{pmatrix} \\ \begin{pmatrix} \Lambda2 \\ \Lambda1 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \Lambda1 \\ \Lambda2 \\ \Lambda3 \\ \Lambda4 \end{pmatrix} & \begin{pmatrix} 0 \\ \Lambda3 \\ \Lambda2 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ \Lambda4 \\ 0 \\ \Lambda2 \end{pmatrix} \\ \begin{pmatrix} \Lambda3 \\ 0 \\ \Lambda1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ \Lambda3 \\ \Lambda2 \\ 0 \end{pmatrix} & \begin{pmatrix} \Lambda1 \\ \Lambda2 \\ \Lambda3 \\ \Lambda4 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ \Lambda4 \\ \Lambda3 \end{pmatrix} \\ \begin{pmatrix} \Lambda4 \\ 0 \\ 0 \\ \Lambda1 \end{pmatrix} & \begin{pmatrix} 0 \\ \Lambda4 \\ 0 \\ \Lambda2 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ \Lambda4 \\ \Lambda3 \end{pmatrix} & \begin{pmatrix} \Lambda1 \\ \Lambda2 \\ \Lambda3 \\ \Lambda4 \end{pmatrix} \end{pmatrix}$$

$\pi^{\rho}_{\mu\nu}$ symmetry is Symmetric[{1, 2, 3}]

```
Λvector = TensorContract[Λform, {1, 2}];
Print[" Λ   =   ", %/4 // MatrixForm]
```

$$\Lambda \;=\; \begin{pmatrix} \Lambda 1 \\ \Lambda 2 \\ \Lambda 3 \\ \Lambda 4 \end{pmatrix}$$

Create $\Gamma^{\rho}_{\mu\nu} = \Pi^{\rho}_{\mu\nu} + \pi^{\rho}_{\mu\nu}$,

```
Γtensor = Πtensor + Λform;
Print["Γ^ρ_μν   = ", Γtensor // MatrixForm]
```

$$\Gamma^{\rho}_{\mu\nu} \;=\; \begin{pmatrix}
\begin{pmatrix} \Lambda 1 \\ \Lambda 2 \\ \Lambda 3 \\ \Lambda 4 \end{pmatrix} & \begin{pmatrix} \Lambda 2 \\ \Lambda 1 \\ \Pi 4 \\ -\Pi 3 \end{pmatrix} & \begin{pmatrix} \Lambda 3 \\ -\Pi 4 \\ \Lambda 1 \\ \Pi 2 \end{pmatrix} & \begin{pmatrix} \Lambda 4 \\ \Pi 3 \\ -\Pi 2 \\ \Lambda 1 \end{pmatrix} \\
\begin{pmatrix} \Lambda 2 \\ \Lambda 1 \\ -\Pi 4 \\ \Pi 3 \end{pmatrix} & \begin{pmatrix} \Lambda 1 \\ \Lambda 2 \\ \Lambda 3 \\ \Lambda 4 \end{pmatrix} & \begin{pmatrix} \Pi 4 \\ \Lambda 3 \\ \Lambda 2 \\ -\Pi 1 \end{pmatrix} & \begin{pmatrix} -\Pi 3 \\ \Lambda 4 \\ \Pi 1 \\ \Lambda 2 \end{pmatrix} \\
\begin{pmatrix} \Lambda 3 \\ \Pi 4 \\ \Lambda 1 \\ -\Pi 2 \end{pmatrix} & \begin{pmatrix} -\Pi 4 \\ \Lambda 3 \\ \Lambda 2 \\ \Pi 1 \end{pmatrix} & \begin{pmatrix} \Lambda 1 \\ \Lambda 2 \\ \Lambda 3 \\ \Lambda 4 \end{pmatrix} & \begin{pmatrix} \Pi 2 \\ -\Pi 1 \\ \Lambda 4 \\ \Lambda 3 \end{pmatrix} \\
\begin{pmatrix} \Lambda 4 \\ -\Pi 3 \\ \Pi 2 \\ \Lambda 1 \end{pmatrix} & \begin{pmatrix} \Pi 3 \\ \Lambda 4 \\ -\Pi 1 \\ \Lambda 2 \end{pmatrix} & \begin{pmatrix} -\Pi 2 \\ \Pi 1 \\ \Lambda 4 \\ \Lambda 3 \end{pmatrix} & \begin{pmatrix} \Lambda 1 \\ \Lambda 2 \\ \Lambda 3 \\ \Lambda 4 \end{pmatrix}
\end{pmatrix}$$

## III. Verifying Symmetry of $\Gamma^{\lambda}_{\nu\gamma}\,\Gamma^{\gamma}_{\lambda\mu}$

Create tensor product of $\Gamma^{\lambda}_{\nu\gamma}$ and $\Gamma^{\gamma}_{\lambda\mu}$ then collapsing the $\gamma$-$\gamma$ indices and the $\lambda$-$\lambda$ indices. Symmetry is then tested and verified.

```
term3 = Apart[TensorContract[TensorProduct [Γtensor, Γtensor], {{1, 5}, {3, 4}}]];
Print["Γ^λ_νγ Γ^γ_λμ is  is ", TensorSymmetry[term3]]
```

$\Gamma^{\lambda}_{\nu\gamma}\Gamma^{\gamma}_{\lambda\mu}$ is  is Symmetric[{1, 2}]

## IV. Proof that $\Gamma^{\lambda}_{\nu\gamma}\,\Gamma^{\gamma}_{\lambda\mu} - \Gamma^{\lambda}_{\mu\gamma}\,\Gamma^{\gamma}_{\lambda\nu} = 0$

```
Apart[Transpose[term3] − term3];
Print["Γ^λ_νγ Γ^γ_λμ − Γ^λ_μγ Γ^γ_λν  = ", %]
```

$$\Gamma^{\lambda}_{\nu\gamma}\Gamma^{\gamma}_{\lambda\mu} - \Gamma^{\lambda}_{\mu\gamma}\Gamma^{\gamma}_{\lambda\nu} \;=\; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

## V. Proof that $\Gamma^{\lambda}_{\nu\gamma}\,\Gamma^{\gamma}_{\lambda\mu} + \Gamma^{\lambda}_{\mu\gamma}\,\Gamma^{\gamma}_{\lambda\nu} = 2\,\Gamma^{\lambda}_{\nu\gamma}\,\Gamma^{\gamma}_{\lambda\mu}$

```
FullSimplify[Expand[term3 + Transpose[term3]]];
Print["Γ^λ_νγ Γ^γ_λμ + Γ^λ_μγ Γ^γ_λν  is ", TensorSymmetry[%]]
Expand[2 term3 − %%];
Print["(Γ^λ_νγ Γ^γ_λμ + Γ^λ_μγ Γ^γ_λν) − 2Γ^λ_νγ Γ^γ_λμ  =  ", %]
```

$\Gamma^{\lambda}_{\nu\gamma}\Gamma^{\gamma}_{\lambda\mu}+\Gamma^{\lambda}_{\mu\gamma}\Gamma^{\gamma}_{\lambda\nu}$ is Symmetric[{1, 2}]

$$(\Gamma^{\lambda}_{\nu\gamma}\Gamma^{\gamma}_{\lambda\mu}+\Gamma^{\lambda}_{\mu\gamma}\Gamma^{\gamma}_{\lambda\nu})-2\Gamma^{\lambda}_{\nu\gamma}\Gamma^{\gamma}_{\lambda\mu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

## VI. Bianchi Identity when Torsion is Totally Antisymmetric

The Bianchi Identity can be written [[9] equation 3.78]

$$D_\mu T^{\kappa}_{\nu\rho} + D_\nu T^{\kappa}_{\rho\mu} + D_\rho T^{\kappa}_{\mu\nu} = R^{\kappa}_{\mu\nu\rho} + R^{\kappa}_{\nu\rho\mu} + R^{\kappa}_{\rho\mu\nu}$$

If we multiply this equation by $\delta^{\rho}_{\kappa}$ and apply total antisymmetry of the torsion, we have

$$D_\alpha T^{\alpha}_{\mu\nu} = R^{x}_{\mu\nu x} + R^{x}_{\nu x\mu} + R^{x}_{x\mu\nu}$$

But we know that

$$R_{\mu\nu} = R^{x}{}_{\nu x\mu} - R^{x}{}_{\mu x\nu} = R^{x}{}_{\nu x\mu} + R^{x}{}_{\mu\nu x} = \left(\partial_\lambda \Gamma^{\lambda}_{\mu\nu} - \partial_\mu \Lambda_\nu + \Lambda_\gamma \Gamma^{\gamma}_{\mu\nu} - \Gamma^{\lambda}_{\mu\gamma} \Gamma^{\gamma}_{\lambda\nu}\right) + \left( \partial_\nu \Lambda_\mu - \partial_\lambda \Gamma^{\lambda}_{\nu\mu} + \Gamma^{\lambda}_{\nu\gamma} \Gamma^{\gamma}_{\lambda\mu} - \Lambda_\gamma \Gamma^{\gamma}_{\nu\mu}\right).$$

This simplifies with regrouping to

$$R_{\mu\nu} = \left(\partial_\lambda \Gamma^{\lambda}_{\mu\nu} - \partial_\lambda \Gamma^{\lambda}_{\nu\mu} + \Lambda_\gamma \Gamma^{\gamma}_{\mu\nu} - \Lambda_\gamma \Gamma^{\gamma}_{\nu\mu}\right) + (\partial_\nu \Lambda_\mu - \partial_\mu \Lambda_\nu) + \left(\Gamma^{\lambda}_{\nu\gamma} \Gamma^{\gamma}_{\lambda\mu} - \Gamma^{\lambda}_{\mu\gamma} \Gamma^{\gamma}_{\lambda\nu}\right).$$

Since

$$\partial_\nu \Lambda_\mu - \partial_\mu \Lambda_\nu = 0$$

and

$$\Gamma^{\lambda}_{\nu\gamma} \Gamma^{\gamma}_{\lambda\mu} - \Gamma^{\lambda}_{\mu\gamma} \Gamma^{\gamma}_{\lambda\nu} = 0$$

we have the Bianchi Identity.

$$R_{\mu\nu} = \mathcal{R}^{(A)}_{\mu\nu} = (\partial_\lambda + \Lambda_\lambda)\, T^{\lambda}_{\mu\nu}.$$

where identification of this curvature is with the antisymmetric part of a reduced curvature.