Ramifications of a Totally Antisymmetric Torsion Tensor - I

A Bridge to Einstein's Theory of General Relativity

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Abstract

In this paper, we show that a totally antisymmetric torsion tensor implies that four unique scalars completely describe the state of torsion on a four-dimensional Riemann- Cartan manifold. These scalars form a vector proportional to the Hodge Dual of the torsion tensor.

The metric connection is shown to consist of a totally antisymmetric component and a symmetric component containing only diagonal elements when the ECE constraint of an antisymmetric metric connection is applied. This symmetric component when summed along the diagonals produces three one-forms which are all assumed equal.

From this, an asymmetric rank two reduced curvature tensor, similar to the Ricci curvature tensor is developed. It is shown to contain a symmetric linear component which can be related to the Einstein general theory of relativity, an antisymmetric component which links to the Einstein, Cartan, Sciama, and Kibble torsional extension of Einstein's earlier work, and a symmetric non-linear curvature term that could be interpreted as the propagation of a non-linear wave.

ECE theory has assumed that constraints imposed by commutator antisymmetry meant that the metric connection was antisymmetric with no non-zero symmetric elements. The question of vanishing diagonal element in the symmetric part of the connection is vague however. When this restriction is relaxed, it allowed a seamless flow or bridge from the general relativity of Einstein to one containing torsion such as the ECE theory. The flow to one of the earliest torsion-curvature relativity theories as provided by Einstein, Cartan, Sciama and Kibble is also demonstrated.

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Introduction

The idea of a vector-based torsion field is not new [1,2]. Socolovsky for example, demonstrated that a totally antisymmetric torsion tensor implies the validity of the equivalence principle of general relativity, and that this tensor field could be used to describe Newtonian gravity. Fabbri [3] developed the geometry for the completely antisymmetric torsion tensor assuming metric compatibility. He demonstrated that this allows writing the metric connection as the sum of a totally antisymmetric connection and a symmetric connection. He also has shown that the Bianchi identity in this limited torsion plus curvature field bears a strong resemblance to the Einstein-Cartan-Sciami-Kibble model of relativity. Socolovsky [1] demonstrated that this was a necessary and sufficient condition for the creation of a local inertial coordinate system at any point in a four-dimensional space.

This paper, an extension to the previous works of the authors [4,5,6], introduces the Hodge Dual of the totally antisymmetric torsion tensor which is shown to be a 4-vector. It is assumed that the reader is familiar with the proofs, based on the antisymmetry of the commutator [11] that the metric connection is antisymmetric except possibly for its diagonal elements. In this paper it is shown that the Levi-Civita connection, the symmetric part of the metric connection, must be purely diagonal.

The Metric Connection for a Totally Antisymmetric Torsion

In this section, properties of a Riemann-Cartan geometry are given when a non-trivial torsion tensor exists, and is totally antisymmetric. The following are assumed: for a four dimensional spacetime, the torsion is non-vanishing and totally antisymmetric, the metric is defined and satisfies the metric compatibility equation, and the metric connection is antisymmetric except for the diagonals which may be non-zero, and is totally diagonally symmetric. This latter restriction is shown to reduce equation complexity. We note that Trautman [[12], equation 12], advises that the transposed connection of a Riemann-Cartan space is metric if and only if the {torsion} tensor is completely antisymmetric. If so, assuming metric compatibility is equivalent to a totally antisymmetric torsion, and vice versa.

Consider the tensor for a totally antisymmetric torsion on a Riemann-Cartan manifold in four dimensions. This class of manifold is one that is both metric compatible, and supports non-zero torsion and also been termed metric hyper-compatible [3,7].

Previously, some ramifications of a totally antisymmetric torsion tensor were presented by the authors [4,5,6] within the framework of the ECE formulation of Cartan geometry. In a four-dimensional Riemann-Cartan spacetime, this antisymmetry is given by [[6], equation 2.54]

$$T^{\rho}{}_{\mu\nu} = -T^{\rho}{}_{\nu\mu}; \quad T^{\rho}{}_{\mu\nu} = -T^{\mu}{}_{\rho\nu}; \quad T^{\rho}{}_{\mu\nu} = -T^{\nu}{}_{\mu\rho}. \tag{1}$$

Applying antisymmetry a second time gives,

$$T^{\rho}{}_{\mu\nu} = T^{\mu}{}_{\nu\rho} = T^{\nu}{}_{\rho\mu}\,. \tag{2}$$

The torsion tensor remains totally antisymmetry when its indices are raised or lowered through the application of the metric tensor because the product of a symmetric tensor and an antisymmetric tensor remains antisymmetric.

All of the the diagonal elements of a totally antisymmetric torsion tensor are zero (summation over repeated indices is not implied in this instance), i.e.

$$T^{\rho}{}_{\mu\mu} = T^{\mu}{}_{\mu\rho} = T^{\mu}{}_{\rho\mu} = 0. \tag{3}$$

Upon examining equations (1) through (3) in a term by term manner, we see that there are four scalars that describe the off-diagonal elements in the torsion tensor. For example, if we examine the case where $\rho = 0$, $\mu = 1$ and $\nu = 2$, we can define a scalar T_3 such that

$$T_3 = T^0{}_{12} = T^1{}_{20} = T^2{}_{01} = -\,T^0{}_{21} = -T^1{}_{02} = -T^2{}_{10} \tag{4}$$

which has magnitude $|T_3|$ and a basis vector associated with the index “3”, an index value differing from all of the index values for the associated torsion tensor elements, in this case 0, 1, and 2. Doing this for each set of values for ρ, μ and ν , results in four components of a covariant 4-vector T_λ defining the totally antisymmetric torsion with its components aligned with each of the unit vectors defining the basis for the torsion tensor.

The Hodge Dual of a rank three totally antisymmetric tensor mapped to a one dimensional sub-manifold is given by [[9], equation 2.114 with p=1 and n=4]

$$T_{\rho\mu\nu} = |g|^{-1/2}\,\epsilon^{\lambda}{}_{\rho\mu\nu}\,T_\lambda \tag{5}$$

where T_λ is a covariant 4-vector. We identify $T_{\rho\mu\nu}$ with the covariant rank three torsion tensor and T_λ with a torsion 4-vector, the Hodge Dual of the torsion tensor.

Rotating the λ index and raising the ρ index results in

$$T^{\rho}{}_{\mu\nu} = |g|^{-1/2}\,g^{\rho\sigma}\,\epsilon_{\sigma\mu\nu}{}^{\lambda}\,T_\lambda = |g|^{-1/2}\,\epsilon^{\rho}{}_{\mu\nu}{}^{\lambda}\,T_\lambda. \tag{6}$$

Taking the Hodge Dual of the torsion tensor returns the 4-vector; taking the Hodge Dual again gives the original tensor (to within a sign) as shown in the appendices to this paper.

$$\tilde{\tilde{T}}^{\rho}{}_{\mu\nu} = \tilde{T}_\lambda = T^{\rho}_{\mu\nu}. \tag{7}$$

When the torsion is totally antisymmetric, the metric compatibility equation is [2,5,6,8]

$$\Gamma^{\rho}_{\nu\mu} + \Gamma^{\rho}_{\mu\nu} = g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \tag{8}$$

which when combined with the definition of torsion in terms of the metric connection, [4,5,6,8] namely,

$$T^{\rho}{}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\nu\mu} \tag{9}$$

gives

$$T^{\rho}{}_{\mu\nu} = 2\,\Gamma^{\rho}_{\mu\nu} - g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})\,. \tag{10}$$

In general, we can write $\Gamma^{\rho}_{\mu\nu}$ as the sum of $\Pi^{\rho}{}_{\mu\nu}$, the antisymmetric portion of the metric connection and $\pi^{\rho}_{\mu\nu}$, the symmetric portion of the metric connection [[3] equation 5], i.e.

$$\Gamma^{\rho}_{\mu\nu} = \Pi^{\rho}{}_{\mu\nu} + \pi^{\rho}_{\mu\nu}, \tag{11}$$

so that substituting this into equation (8) gives

$$\pi^{\rho}_{\mu\nu} = \frac{1}{2}\,g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \;\; for\, all\, \rho,\; \mu,\; \nu, \tag{12}$$

and

$$T^{\rho}{}_{\mu\nu} = 2\,\Pi^{\rho}{}_{\mu\nu} \;\; for\, all\, \rho,\; \mu,\; \nu. \tag{13}$$

Equation (13) demonstrates that $\Pi^{\rho}{}_{\mu\nu}$, the antisymmetric portion of the metric connection, is also a tensor, and is totally antisymmetric. Thus we can write, applying equation (7), (9), and (11)

$$\tilde{T}^{\rho}{}_{\mu\nu} = 2\,\tilde{\Pi}^{\rho}{}_{\mu\nu} = 2\,\Pi_\lambda, \tag{14}$$

where Π_λ is a 4-vector, the Hodge Dual of the antisymmetric part of the metric connection.

As mentioned earlier, it is assumed that the reader is familiar with the proofs, based on the antisymmetry of the commutator [11], that the metric connection is antisymmetric except for possible non-zero diagonal elements. Three diagonals exist in the metric connection, $\pi^{\rho}_{\mu\rho}$, $\pi^{\rho}_{\rho\mu}$, and $\pi^{\rho}_{\mu\mu}$. With $T^{\rho}{}_{\mu\nu}$ and $\Pi^{\rho}{}_{\mu\nu}$ being totally antisymmetric, $\pi^{\rho}_{\mu\nu}$ must vanish whenever $\rho \neq \mu \neq \nu$, so that $\pi^{\rho}_{\mu\nu}$ can be non-zero only when any two (or more) of the indices are identical. The only symmetric non-zero components of $\pi^{\rho}_{\mu\nu}$ remaining are then (no summation implied) $\pi^{\rho}_{\rho\nu}$, $\pi^{\rho}_{\mu\rho}$, $\pi^{\rho}_{\mu\mu}$ (and $\pi^{\mu}_{\mu\mu}$), i.e. $\pi^{\rho}_{\mu\nu}$ is purely diagonal.

In an earlier publication [6], the concept of "shear" of the spacetime was introduced and assumed to be totally symmetric, with diagonal entries only, which as a result of equation (11) gives

$$S^{\rho}{}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} + \Gamma^{\rho}_{\nu\mu} = 2\,\pi^{\rho}_{\mu\nu} \tag{15}$$

Shear being totally or purely diagonal then results in the symmetric part of the connection being totally diagonal. The shear was also shown [[6], equation 2.69] to be totally symmetric which for this situation where only diagonal elements can be non-zero, means that

$$S_{\rho\rho\mu} = S_{\rho\mu\rho} = S_{\mu\rho\rho} \tag{16}$$

which requires

$$\pi_{\rho\rho\mu} = \pi_{\rho\mu\rho} = \pi_{\mu\rho\rho} \tag{17}$$

From equation (12), we see that $\pi^{\rho}_{\mu\nu}$ is symmetric in the lower two indices and that we can form three diagonal sums, two of which are identical. That is, for the symmetric part,

$$\Gamma^{\rho}_{\mu\rho} = \pi^{\rho}_{\mu\rho} = \pi^{\rho}_{\rho\mu} = \pi^{\rho}_{\mu\nu}\,\delta^{\nu}_{\rho} = \frac{1}{2}\, g^{\rho\sigma}(\partial_\mu g_{\rho\sigma} + \partial_\rho g_{\sigma\mu} - \partial_\sigma g_{\mu\rho}) \;. \tag{18}$$

Noting that [[2] equation 3.4.7 and 3.4.9] i.e.

$$\frac{1}{2}\, g^{\rho\mu}\,\partial_\nu g_{\rho\mu} = \partial_\nu \ln\left(\sqrt{|g|}\right) = \frac{1}{2}\,\partial_\nu \ln|g|, \tag{19}$$

equation (18)becomes

$$\Lambda_\mu = \pi^{\rho}_{\mu\rho} = \partial_\mu \ln\sqrt{|g|} + \frac{1}{2} g^{\rho\sigma}\left(\partial_\rho g_{\sigma\mu} - \partial_\sigma g_{\mu\rho}\right) = \partial_\mu \ln\sqrt{|g|}\,, \tag{20}$$

where we have used $g^{\rho\sigma}\,(\partial_\rho g_{\sigma\mu} - \partial_\sigma g_{\mu\rho}) = 0$, when summed over both ρ and σ.

We also have from equation (12), collapsing the lower two indices, that

$$\begin{aligned} X^\rho = \pi^{\rho}_{\mu\nu}\, g^{\mu\nu} &= \frac{1}{2}\, g^{\rho\sigma}\, g^{\mu\nu}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) = \\ &\frac{1}{2}\, g^{\rho\sigma}\left(g^{\mu\nu}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu}) - \partial_\sigma \ln\sqrt{|g|}\right) \end{aligned} \tag{21}$$

Pulling down the ρ, σ indices, we have

$$X_\rho = \frac{1}{2}\, g^{\mu\nu}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu}) - \frac{1}{2}\,\partial_\rho \ln\sqrt{|g|} = g^{\mu\nu}\,\partial_\mu g_{\nu\rho} - \frac{1}{2}\,\partial_\rho \ln\sqrt{|g|}\,. \tag{22}$$

Noting equations (20) and (22), we see that

$$X_\rho + \Lambda_\rho = g^{\mu\nu}\,\partial_\mu g_{\nu\rho} \tag{23}$$

If we let $X_\rho = \Lambda_\rho$, the assumption of total diagonal symmetry, then

$$X_\rho = \Lambda_\rho = \frac{1}{2}\, g^{\mu\nu}\,\partial_\mu g_{\nu\rho} = \partial_\rho \ln\sqrt{|g|}\,. \tag{24}$$

This assumption reduces equation complexity by a significant amount. If warranted, it will be relaxed in future publications.

So far, we have established the following, given a four dimensional spacetime with a non-vanishing totally antisymmetric torsion, a metric that satisfies the metric compatibility equation, and a metric connection that is antisymmetric except for the diagonals which may be non-zero, we have that

- the torsion tensor can be represented by a vector in four dimensions. This vector is the Hodge Dual of the torsion tensor,
- the antisymmetric portion of the metric connection is totally antisymmetric, and

• the symmetric portion of the metric connection is assumed to be purely diagonal. This is represented by three vectors, all assumed equal, with values given by by the derivative of a scalar.

Curvature When Torsion is Totally Antisymmetric

The rank four curvature tensor in four dimensions on the base manifold and its cyclically rotated equivalents are given by the following [[[5] equation 1.24 ; UFT 88, equation 2],

$$R^{\lambda}{}_{\rho\mu\nu} = \partial_\mu \Gamma^{\lambda}_{\nu\rho} - \partial_\nu \Gamma^{\lambda}_{\mu\rho} + \Gamma^{\lambda}_{\mu\gamma} \Gamma^{\gamma}_{\nu\rho} - \Gamma^{\lambda}_{\nu\gamma} \Gamma^{\gamma}_{\mu\rho} \quad (25)$$

$$R^{\lambda}{}_{\mu\nu\rho} = \partial_\nu \Gamma^{\lambda}_{\rho\mu} - \partial_\rho \Gamma^{\lambda}_{\nu\mu} + \Gamma^{\lambda}_{\nu\gamma} \Gamma^{\gamma}_{\rho\mu} - \Gamma^{\lambda}_{\rho\gamma} \Gamma^{\gamma}_{\nu\mu} \quad (26)$$

$$R^{\lambda}{}_{\nu\rho\mu} = \partial_\rho \Gamma^{\lambda}_{\mu\nu} - \partial_\mu \Gamma^{\lambda}_{\rho\nu} + \Gamma^{\lambda}_{\rho\gamma} \Gamma^{\gamma}_{\mu\nu} - \Gamma^{\lambda}_{\mu\gamma} \Gamma^{\gamma}_{\rho\nu} . \quad (27)$$

Three reduced or contracted curvatures can be generated by collapsing the tensor using the upper index and one of the lower indices. The first reduced curvature, $R^{x}{}_{x\mu\nu}$, (the symbol x is a placeholder only) can be derived from the curvature tensor in equation (25) by contracting it with δ^{ρ}_{λ}

$$R^{x}{}_{x\mu\nu} = R^{\lambda}{}_{\rho\mu\nu}\, \delta^{\rho}_{\lambda} = \left(\partial_\mu \Gamma^{\lambda}_{\nu\rho} - \partial_\nu \Gamma^{\lambda}_{\mu\rho}\right) \delta^{\rho}_{\lambda} + \left(\Gamma^{\lambda}_{\mu\gamma} \Gamma^{\gamma}_{\nu\rho} - \Gamma^{\lambda}_{\nu\gamma} \Gamma^{\gamma}_{\mu\rho}\right) \delta^{\rho}_{\lambda} \quad (28)$$

If we interchange the dummy variables associated with the summations, we notice that the difference $\Gamma^{\lambda}_{\mu\gamma} \Gamma^{\gamma}_{\nu\lambda}$ - $\Gamma^{\lambda}_{\nu\gamma} \Gamma^{\gamma}_{\mu\lambda}$ in equation (28) vanishes.

Expanding this, using equation (20), we have

$$\begin{gathered} R^{x}{}_{x\mu\nu} = \left(\partial_\mu \Gamma^{\lambda}_{\nu\rho} - \partial_\nu \Gamma^{\lambda}_{\mu\rho}\right) \delta^{\rho}_{\lambda} = \\ \partial_\mu \pi^{\lambda}_{\nu\lambda} - \partial_\nu \pi^{\lambda}_{\mu\lambda} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu = \partial_\mu \partial_\nu \ln \sqrt{|g|} - \partial_\nu \partial_\mu \ln \sqrt{|g|} = 0. \end{gathered} \quad (29)$$

Another reduced curvature is obtained by collapsing the λ, ρ indices in equation (26), again noting that the curvature is antisymmetric in the last two indices, we have, noting $\Gamma^{\lambda}_{\lambda\gamma}$=$\Lambda_\gamma$, for second the reduced curvature tensor,

$$R^{x}{}_{\mu\nu x} = -R^{x}{}_{\mu x\nu} = \partial_\nu \Lambda_\mu - \partial_\lambda \Gamma^{\lambda}_{\nu\mu} + \Gamma^{\lambda}_{\nu\gamma} \Gamma^{\gamma}_{\lambda\mu} - \Lambda_\gamma \Gamma^{\gamma}_{\nu\mu} . \quad (30)$$

A third reduced curvature is obtained by multiplying equation (27) by δ^{ρ}_{λ} noting again that $\Gamma^{\lambda}_{\lambda\gamma}$=$\Lambda_\gamma$,

$$R^{x}{}_{\nu x\mu} = R^{\lambda}{}_{\nu\rho\mu}\, \delta^{\rho}_{\lambda} = \partial_\lambda \Gamma^{\lambda}_{\mu\nu} - \partial_\mu \Lambda_\nu + \Lambda_\gamma \Gamma^{\gamma}_{\mu\nu} - \Gamma^{\lambda}_{\mu\gamma} \Gamma^{\gamma}_{\lambda\nu} . \quad (31)$$

In the appendix it is shown that $\Gamma^{\lambda}_{\nu\gamma} \Gamma^{\gamma}_{\lambda\mu}$ is symmetric in μ and ν, when the symmetric part of the connection is totally symmetric. In this case,

$$\Gamma^{\lambda}_{\nu\gamma}\,\Gamma^{\gamma}_{\lambda\mu} + \Gamma^{\lambda}_{\mu\gamma}\,\Gamma^{\gamma}_{\lambda\nu} = 2\,\Gamma^{\lambda}_{\nu\gamma}\,\Gamma^{\gamma}_{\lambda\mu}\,. \tag{32}$$

If we define an antisymmetric curvature by

$$\begin{gathered}\mathcal{R}^{(A)}_{\mu\nu} = -R^{x}{}_{\mu x\nu} + R^{x}{}_{\nu x\mu} = \\ \left(\partial_\nu \Lambda_\mu - \partial_\lambda \Gamma^{\lambda}_{\nu\mu} + \Gamma^{\lambda}_{\nu\gamma}\,\Gamma^{\gamma}_{\lambda\mu} - \Lambda_\gamma\,\Gamma^{\gamma}_{\nu\mu}\right) + \left(\partial_\lambda \Gamma^{\lambda}_{\mu\nu} - \partial_\mu \Lambda_\nu + \Lambda_\gamma\,\Gamma^{\gamma}_{\mu\nu} - \Gamma^{\lambda}_{\mu\gamma}\,\Gamma^{\gamma}_{\lambda\nu}\right),\end{gathered} \tag{33}$$

then using equations (24), (29) through (32) this becomes

$$\mathcal{R}^{(A)}_{\mu\nu} = \partial_\lambda\left(\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}\right) + \Lambda_\lambda\left(\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}\right). \tag{34}$$

We have, from equations (9) and (13), this is

$$\mathcal{R}^{(A)}_{\mu\nu} = (\partial_\lambda + \Lambda_\lambda)\; T^{\lambda}_{\mu\nu} = 2\,(\partial_\lambda + \Lambda_\lambda)\,\Pi^{\lambda}_{\mu\nu}. \tag{35}$$

This is the first Bianchi identity under the assumptions given in this paper.

We can also define a symmetric reduced curvature tensor $\mathcal{R}^{(S)}_{\mu\nu}$ by subtracting equation (30) from equation (31),

$$\begin{gathered}\mathcal{R}^{(S)}_{\mu\nu} = R^{x}{}_{\mu x\nu} + R^{x}{}_{\nu x\mu} = \\ -2\,\partial_\mu \partial_\nu \ln\sqrt{|g|} + \Lambda_\gamma(\Gamma^{\gamma}_{\mu\nu} + \Gamma^{\gamma}_{\nu\mu}) + \partial_\gamma(\Gamma^{\gamma}_{\mu\nu} + \Gamma^{\gamma}_{\nu\mu}) - \left(\Gamma^{\lambda}_{\nu\gamma}\,\Gamma^{\gamma}_{\lambda\mu} + \Gamma^{\lambda}_{\mu\gamma}\,\Gamma^{\gamma}_{\lambda\nu}\right)\quad .\end{gathered} \tag{36}$$

Using the spacetime shear of equation (15), allows equation (36) to be written

$$\mathcal{R}^{(S)}_{\mu\nu} = (\partial_\gamma + \Lambda_\gamma)\; S^{\gamma}_{\mu\nu} + \mathcal{R}^{(\mathrm{NL})}_{\mu\nu} \tag{37}$$

where, introducing

$$\phi = \ln\sqrt{|g|} \tag{38}$$

we have, noting equation (32)

$$\mathcal{R}^{(\mathrm{NL})}_{\mu\nu} = -2\,\partial_\mu\partial_\nu\phi - \left(\Gamma^{\lambda}_{\nu\gamma}\,\Gamma^{\gamma}_{\lambda\mu} + \Gamma^{\lambda}_{\mu\gamma}\,\Gamma^{\gamma}_{\lambda\nu}\right) = -2\,\partial_\mu\partial_\nu\phi - 2\,\Gamma^{\lambda}_{\nu\gamma}\,\Gamma^{\gamma}_{\lambda\mu}\,. \tag{39}$$

This is a symmetric reduced curvature term $\mathcal{R}^{(\mathrm{NL})}_{\mu\nu}$ which contains all of the non-linear terms in the curvature equations. This allows us to write the so-called symmetric Einstein curvature tensor for this geometry as

$$G_{\mu\nu} = \mathcal{R}^{(S)}_{\mu\nu} - \frac{\mathcal{R}}{2}\,g_{\mu\nu} = (\partial_\gamma + \Lambda_\gamma)\; S^{\gamma}_{\mu\nu} - \frac{1}{2}\,(\partial_\gamma + \Lambda_\gamma)\; S^{\gamma}_{\alpha\beta}\,g^{\alpha\beta}\,g_{\mu\nu} + \left(\mathcal{R}^{(\mathrm{NL})}_{\mu\nu} - \frac{1}{2}\,\mathcal{R}^{(\mathrm{NL})}\,g_{\mu\nu}\right) \tag{40}$$

where the scalar curvature, $\mathcal{R}$, the trace of the reduced curvature from the above equation is given by

$$\mathcal{R} = \mathcal{R}^{(S)}_{\mu\nu}\,g^{\mu\nu} = \left((\partial_\gamma + \Lambda_\gamma)\; S^{\gamma}_{\alpha\beta}\right)\,g^{\alpha\beta} + \mathcal{R}^{(\mathrm{NL})} \tag{41}$$

where the non-linear scalar curvature is

$$\mathcal{R}^{(\mathrm{NL})} = \mathcal{R}^{(\mathrm{NL})}_{\mu\nu}\, g^{\mu\nu} = -\,2\,\Gamma^{\lambda}_{\nu\gamma}\,\Gamma^{\gamma}_{\lambda\mu}\,g^{\mu\nu} - 2\,\Box\phi\,. \tag{42}$$

If we neglect the non-linear curvature $\mathcal{R}^{(\mathrm{NL})}_{\mu\nu}$ and its scalar curvature $\mathcal{R}^{(\mathrm{NL})}_{\mu\nu}\, g^{\mu\nu}$, the Einstein curvature tensor becomes

$$G_{\mu\nu} = \mathcal{R}^{(S)}_{\mu\nu} - \frac{\mathcal{R}}{2}\, g_{\mu\nu} = (\partial_\gamma + \Lambda_\gamma)\; S^{\gamma}_{\mu\nu} - \frac{1}{2}\,(\partial_\gamma + \Lambda_\gamma)\; S^{\gamma}_{\alpha\beta}\, g^{\alpha\beta}\, g_{\mu\nu} \tag{43}$$

This assumption generates an independent non-linear equation given by

$$\left(\mathcal{R}^{(\mathrm{NL})}_{\mu\nu} - \frac{1}{2}\,\mathcal{R}^{(\mathrm{NL})}\, g_{\mu\nu}\right) = 0 \tag{44}$$

or

$$\partial_\mu\,\partial_\nu\,\phi + \Gamma^{\lambda}_{\nu\gamma}\,\Gamma^{\gamma}_{\lambda\mu} + \left(\Gamma^{\lambda}_{\beta\gamma}\,\Gamma^{\gamma}_{\lambda\alpha}\, g^{\alpha\beta} + \Box\phi\right) g_{\mu\nu} = 0. \tag{45}$$

The wave nature of this equation is noted, but is the subject of another paper.

General Relativity of Einstein, Cartan, Sciama , and Kibble

Einstein’s equations for general relativity [[8] equation 4.52] are for a symmetric Ricci curvature $\mathcal{R}_{\mu\nu}$ given by

$$\mathcal{R}_{\mu\nu} - \frac{\mathcal{R}}{2}\, g_{\mu\nu} = 8\,\pi G\, \tau_{\mu\nu} = G_{\mu\nu} \tag{46}$$

where $\tau_{\mu\nu}$ is the Einstein stress-energy tensor and G is the gravitational constant.

Substituting equation (43) into (46) we see

$$G_{\mu\nu} = (\partial_\gamma + \Lambda_\gamma)\; S^{\gamma}_{\mu\nu} - \frac{1}{2}\left((\partial_\gamma + \Lambda_\gamma)\; S^{\gamma}_{\alpha\beta}\right) g^{\alpha\beta}\, g_{\mu\nu} = 8\,\pi G\, \tau_{\mu\nu} \tag{47}$$

We note that any coupling to the torsion field has been removed by neglecting the non-linear curvature. It should be pointed out that the assumption of zero torsion was not required.

Trautman [[12], equations 23 and 24] summarizes the Einstein, Cartan, Sciama, and Kibble equations of general relativity, perhaps the simplest extension of Einstein’s relativity to include torsion, as two equations, the first being the standard Einstein equation, as given by equation (46) above, and

$$T^{\rho}_{\mu\nu} + \delta^{\rho}_{\mu}\, T^{\sigma}_{\nu\sigma} + \delta^{\rho}_{\nu}\, T^{\sigma}_{\mu\sigma} = 8\,\pi\, s^{\rho}{}_{\mu\nu} \tag{48}$$

where $s^{\rho}{}_{\mu\nu}$ is the spin energy density, and $T^{\rho}_{\mu\nu}$ is the torsion tensor. For the case of a totally antisymmetric torsion, equation (48) becomes

$$T^{\rho}_{\mu\nu} = 8\,\pi\, s^{\rho}{}_{\mu\nu}. \tag{49}$$

If we apply the operator $(\partial_\lambda + \Lambda_\lambda)$ to this equation, we have

$$(\partial_\lambda + \Lambda_\lambda)\; T^{\rho}_{\mu\nu} = 8\,\pi\;(\partial_\lambda + \Lambda_\lambda)\; s^{\rho}{}_{\mu\nu} \;= \mathcal{R}^{(A)}_{\mu\nu} \qquad (50)$$

which from equation (35) we recognize the antisymmetric component of curvature $\mathcal{R}^{(A)}_{\mu\nu}$. This equation is not directly coupled to the curvature equation (47) except through the vector Λ_λ. Direct torsion-curvature coupling disappeared when the non-linear terms were neglected.

Discussion and Conclusions

Equations (47) and (50) provide the bridge from the general relativity of Einstein, and later, Einstein, Cartan, Sciama and Kibble to a general geometry with torsion and curvature which is the basis of the more general ECE general relativity. The path which leads to the ECE theory which is based on Cartan geometry is the subject of another paper however, as is the wave nature of the neglected non-linear curvature terms.

In this paper we have demonstrated that a totally antisymmetric torsion tensor implies that four unique scalars completely describe the state of torsion on a four-dimensional Riemann- Cartan manifold. These scalars are in vector form expressed by the Hodge Dual of the torsion tensor.

The metric connection was shown to consist of at most, a totally antisymmetric component and a symmetric component consisting of only diagonal elements. The symmetric component was summed along the diagonals to produce three one-forms of which all three were assumed equal. This allowed the representation of the symmetric component of the metric connection in terms of a single four-vector also.

If one generates the sum of the reduced curvature tensor and half the scalar curvature multiplied by the metric, the resulting equation shown to reduce to the Einstein equation of general relativity. ECE theory has assumed that constraints imposed by commutator antisymmetry meant that the metric connection did not contain any symmetric elements. When this assumption was relaxed, it allowed a seamless flow or bridge from the general relativity of Einstein to one containing torsion at a level higher than a slight perturbation, such as the ECE theory. The flow to one of the earliest torsion-curvature relativity theories as provided by Einstein, Cartan, Sciama and Kibble was also demonstrated.

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