99(2): Proof of the Jacobi Identity

The Jacobi identity is important both for geometry and field theory and is proven as follows by expanding the commutators:

$$[[D_\lambda, D_\rho], D_\sigma] + [[D_\rho, D_\sigma], D_\lambda] + [[D_\sigma, D_\lambda], D_\rho] := 0 \quad -(1)$$

$$= (D_\lambda D_\rho - D_\rho D_\lambda) D_\sigma - D_\sigma (D_\lambda D_\rho - D_\rho D_\lambda)$$
$$+ (D_\rho D_\sigma - D_\sigma D_\rho) D_\lambda - D_\lambda (D_\rho D_\sigma - D_\sigma D_\rho)$$
$$+ (D_\sigma D_\lambda - D_\lambda D_\sigma) D_\rho - D_\rho (D_\sigma D_\lambda - D_\lambda D_\sigma)$$

$$= D_\lambda D_\rho D_\sigma - D_\rho D_\lambda D_\sigma - D_\sigma D_\lambda D_\rho + D_\sigma D_\rho D_\lambda$$
$$+ D_\rho D_\sigma D_\lambda - D_\sigma D_\rho D_\lambda - D_\lambda D_\rho D_\sigma + D_\lambda D_\sigma D_\rho$$
$$+ D_\sigma D_\lambda D_\rho - D_\lambda D_\sigma D_\rho - D_\rho D_\sigma D_\lambda + D_\rho D_\lambda D_\sigma$$

$$= 0 \quad , \qquad \text{Q.E.D.}$$

The Jacobi identity is used in field theory to define the field through the commutator of covariant derivatives. For example in Ryder's eq.

2) (3.167):

$$[D_\mu, D_\nu] = -ig\, G_{\mu\nu} \quad -(2)$$

where g is a constant and where $G_{\mu\nu}$ is the field.

In Ryder's notation:

$$[D_\mu, D_\nu] = [\partial_\mu - igA_\mu, \partial_\nu - igA_\nu]$$

$$= -ig\left(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]\right) \quad -(3)$$

and this has been extensively developed in electrodynamics. In Ryder's eq. (3.173):

$$D_\rho G_{\mu\nu} + D_\mu G_{\nu\rho} + D_\nu G_{\rho\mu} = 0 \quad -(4)$$

which in the notation of differential geometry is:

$$D \wedge G = 0. \quad -(5)$$

Eq. (5) has an obvious similarity with the Bianchi identity of differential geometry:

$$D \wedge T^a := R^a{}_b \wedge q^b \quad -(6)$$

and as shown in paper 88 there is only one Bianchi identity, there is only one Jacobi identity. o.(3)

which is the notation of differential geometry is:

$$\boxed{D \wedge [D_\mu, D_\nu] := 0} \quad \text{— (7)}$$

There is a relation therefore between the Bia[nchi] identity (6) of differential geometry and the Jacobi identity (7) between operators.

In the absence of torsion:

$$[D_\mu, D_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma \quad \text{— (8)}$$

and

$$R^a{}_b \wedge q^b$$
$$= R^\rho{}_{\sigma\mu\nu} + R^\rho{}_{\nu\sigma\mu} + R^\rho{}_{\mu\nu\sigma} \quad \text{— (}$$
$$= 0$$

where:

$$[D_\sigma, D_\mu] V^\rho = R^\rho{}_{\nu\sigma\mu} V^\nu \quad \text{— (10)}$$
$$[D_\nu, D_\sigma] V^\rho = R^\rho{}_{\mu\nu\sigma} V^\mu \quad \text{— (11)}$$

so

$$\left([D_\mu, D_\nu] + [D_\sigma, D_\mu] + [D_\nu, D_\sigma]\right) V^\rho$$
$$= R^\rho{}_{\sigma\mu\nu} V^\sigma + R^\rho{}_{\nu\sigma\mu} V^\nu + R^\rho{}_{\mu\nu\sigma} V^\mu$$

(12)

4) Now write eq. (12) as:

$$\left([D_\mu, D_\nu] + [D_\sigma, D_\mu] + [D_\nu, D_\sigma] \right) V^\rho$$
$$= \left(R^\rho{}_{\rho\mu\nu} + R^\rho{}_{\rho\sigma\mu} + R^\rho{}_{\rho\nu\sigma} \right) V^\rho \quad -(13)$$

to obtain a new generator relation. Here:

$$R_{\mu\nu} := R^\rho{}_{\rho\mu\nu} \quad -(14)$$
$$R_{\sigma\mu} := R^\rho{}_{\rho\sigma\mu} \quad -(15)$$
$$R_{\nu\sigma} := R^\rho{}_{\rho\nu\sigma} \quad -(16)$$

are antisymmetric:

$$\boxed{R_{\mu\nu} = -R_{\nu\mu}} \quad -(17)$$

and so on

In the presence of torsion: $\quad -(18)$

$$[D_\mu, D_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda{}_{\mu\nu} D_\lambda V^\rho$$

and eq. (18) can be extended. In the notation of differential geometry eq. (18) is:

$$(D \wedge D) V^\rho = R V^\rho \quad -(19)$$

or:

$$\boxed{D \wedge D = R} \quad -(20)$$

5)

where:

$$D \wedge D := D_\mu D_\nu - D_\nu D_\mu + D_\sigma D_\mu - D_\mu D_\sigma + D_\nu D_\sigma - D_\sigma D_\nu \quad -(21)$$

and $R := R_{\mu\nu} + R_{\sigma\mu} + R_{\nu\sigma}$ $\quad -(22)$

Eq (20) is a new equation of differential geometry. It emphasizes that the curvature form is the wedge product of covariant derivatives in the absence of torsion

Ryder identifies the Jacobi identity with the conventional second Bianchi identity:

$$D_\rho R^{\kappa}{}_{\lambda\mu\nu} + D_\mu R^{\kappa}{}_{\lambda\rho\mu} + D_\nu R^{\kappa}{}_{\lambda\rho\mu} = 0 \quad -(23)$$

but as in paper 88 it is now known that eq. (23) is incomplete. However the Jacobi identity is exact. To complete eq. (23) needs consideration of torsion.