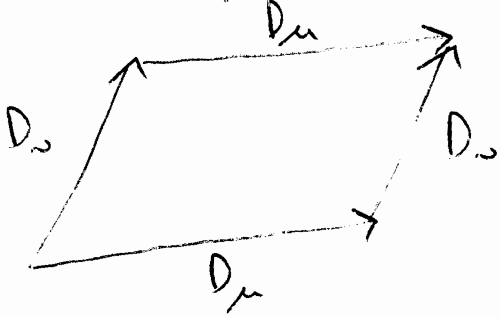
99(1): The Fundamental Origin of Curvature and Torsion

The origin of such tensors is parallel transport around a closed loop:

$$\delta V^{\rho} = (\delta a)(\delta b) A^{\nu} B^{\mu} R^{\rho}{}_{\sigma\mu\nu} V^{\sigma} \quad -(1)$$

which can be represented as a commutator of covariant derivatives. The covariant derivative of a tensor in a given direction measures how much the tensor changes relative to what it would have been if it had been parallel transported. The covariant derivative of a tensor in a direction along which it is parallel transported is zero. The commutator of two covariant derivatives measures the difference between parallel transporting the tensor one way and then the other, versus the opposite ordering.



The same idea is used in field theory, (e.g. L.H. Ryder "Quantum Field Theory" (CUP 1996)) The fundamental origin of such curvature and torsion is therefore the operator:

$$[D_\mu, D_\nu] = D_\mu D_\nu - D_\nu D_\mu \quad -(2)$$

which is a commutator. Such commutators are

2)

Well known is the theory of <u>rotation generators</u> and <u>angular momentum</u>. They appear throughout quantum mechanics. In differential geometry the wedge product of two one-forms A_ν and B_μ is also a commutator

$$A \wedge B = A_\nu B_\mu - A_\mu B_\nu . \quad -(3)$$

The reason why eq (2) is not zero is that the commutator <u>operates</u> on a vector or tensor. In quantum mechanics it operates on the wave-function. In eq (3) A and B are one-forms and not operators.

<u>The Riemann and Torsion Tensors</u>

These are defined by:

$$[D_\mu, D_\nu] V^\rho = D_\mu (D_\nu V^\rho) - D_\nu (D_\mu V^\rho) \quad -(3)$$

where V^ρ is a four-vector in a manifold with curvature and torsion. So on the right hand side of eq. (3) the covariant derivatives operate on rank two tensors $D_\nu V^\rho$ and $D_\mu V^\rho$. The rule for this is:

$$[D_\mu, D_\nu] V^\rho = \partial_\mu (D_\nu V^\rho) - \Gamma^\lambda_{\mu\nu} D_\lambda V^\rho + \Gamma^\rho_{\mu\sigma} D_\nu V^\sigma$$
$$- \partial_\nu (D_\mu V^\rho) - \Gamma^\lambda_{\nu\mu} D_\lambda V^\rho + \Gamma^\rho_{\nu\sigma} D_\mu V^\sigma \quad -(4)$$

3) The covariant derivatives are:

$$\left.\begin{aligned} D_\nu \bar{V}^\rho &= \partial_\nu \bar{V}^\rho + \Gamma^\rho_{\nu\lambda}\bar{V}^\lambda \\ D_\lambda \bar{V}^\rho &= \partial_\lambda \bar{V}^\rho + \Gamma^\rho_{\lambda\sigma}\bar{V}^\sigma \\ D_\nu \bar{V}^\sigma &= \partial_\nu \bar{V}^\sigma + \Gamma^\sigma_{\nu\lambda}\bar{V}^\lambda \end{aligned}\right\} \quad \text{— (5)}$$

So:

$$\begin{aligned} \partial_\mu\left(D_\nu \bar{V}^\rho\right) &= \partial_\mu\partial_\nu \bar{V}^\rho + \left(\partial_\mu \Gamma^\rho_{\nu\lambda}\right)\bar{V}^\lambda + \Gamma^\rho_{\nu\lambda}\partial_\mu \bar{V}^\lambda \\ &= \partial_\mu\partial_\nu \bar{V}^\rho + \left(\partial_\mu \Gamma^\rho_{\nu\sigma}\right)\bar{V}^\sigma + \Gamma^\rho_{\nu\sigma}\partial_\mu \bar{V}^\sigma \end{aligned} \quad \text{— (6)}$$

by re-arranging dummy indices: $\lambda \to \sigma$. — (7)

So:

$$\begin{aligned} [D_\mu, D_\nu]\bar{V}^\rho = {} & \partial_\mu\partial_\nu \bar{V}^\rho + \left(\partial_\mu \Gamma^\rho_{\nu\sigma}\right)\bar{V}^\sigma + \Gamma^\rho_{\nu\sigma}\partial_\mu \bar{V}^\sigma \\ & - \Gamma^\lambda_{\mu\nu}\partial_\lambda \bar{V}^\rho - \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\lambda\sigma}\bar{V}^\sigma \\ & + \Gamma^\rho_{\mu\sigma}\partial_\nu \bar{V}^\sigma + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\lambda}\bar{V}^\lambda \\ & - \partial_\nu\partial_\mu \bar{V}^\rho - \left(\partial_\nu \Gamma^\rho_{\mu\sigma}\right)\bar{V}^\sigma - \Gamma^\rho_{\mu\sigma}\partial_\nu \bar{V}^\sigma \\ & + \Gamma^\lambda_{\nu\mu}\partial_\lambda \bar{V}^\rho + \Gamma^\lambda_{\nu\mu}\Gamma^\rho_{\lambda\sigma}\bar{V}^\sigma \\ & - \Gamma^\rho_{\nu\sigma}\partial_\mu \bar{V}^\sigma - \Gamma^\rho_{\nu\sigma}\Gamma^\sigma_{\mu\lambda}\bar{V}^\lambda \end{aligned} \quad \text{— (8)}$$

4)

$$= \left(\partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \right)$$

$$- \left(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \right)\left(\partial_\lambda \bar{V}^\rho + \Gamma^\rho_{\lambda\sigma}\bar{V}^\sigma \right)$$

$$= R^\rho{}_{\sigma\mu\nu}\bar{V}^\sigma - T^\lambda_{\mu\nu} D_\lambda \bar{V}^\rho \quad \text{— (9)}$$

$$\boxed{[D_\mu, D_\nu]\bar{V}^\rho = R^\rho{}_{\sigma\mu\nu}\bar{V}^\sigma - T^\lambda_{\mu\nu} D_\lambda \bar{V}^\rho} \quad \text{— (10)}$$

Here $R^\rho{}_{\sigma\mu\nu}$ is the Riemann tensor:

$$R^\rho{}_{\sigma\mu\nu} = - R^\rho{}_{\sigma\nu\mu} \quad \text{— (11)}$$

and $T^\lambda_{\mu\nu}$ is the torsion tensor.

$$T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \quad \text{— (12)}$$

$$= - T^\lambda_{\nu\mu}$$

In flat or Minkowski space-time:

$$[\partial_\mu, \partial_\nu] = 0 \quad \text{— (13)}$$

$$R^\rho{}_{\sigma\mu\nu} = 0 \quad \text{— (14)}$$

$$T^\lambda_{\mu\nu} = 0. \quad \text{— (15)}$$

5)

The Riemann and torsion tensors are constructed from connection and are true for any connection, whether metric compatible or torsion free. Although connection is non-tensorial the Riemann and torsion tensors are true, generally covariant, tensors.

For a tensor of any rank:

$$[D_\rho, D_\sigma] X^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} = - T^\lambda{}_{\rho\sigma} D_\lambda X^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}$$

$$+ R^{\mu_1}{}_{\lambda\rho\sigma} X^{\lambda \mu_2 \dots \mu_k}{}_{\nu_1 \dots \nu_l} + R^{\mu_2}{}_{\lambda\rho\sigma} X^{\mu_1 \lambda \dots \mu_k}{}_{\nu_1 \dots \nu_l} + \cdots$$

$$- R^\lambda{}_{\nu_1\rho\sigma} X^{\mu_1 \dots \mu_k}{}_{\lambda \nu_2 \dots \nu_l} - R^\lambda{}_{\nu_2\rho\sigma} X^{\mu_1 \dots \mu_k}{}_{\nu_1 \lambda \dots \nu_l} - \cdots$$

— (16)

The commutator of two vector fields X and Y is a third vector field with components:

$$[X, Y]^\mu = X^\lambda \partial_\lambda Y^\mu - Y^\lambda \partial_\lambda X^\mu$$

— (17)

The torsion and Riemann tensors can be thought of as [mu]lti-linear maps. The torsion is a map from [tw]o vector fields to a third vector field:

$$T(X, Y) = D_X Y - D_Y X - [X, Y]$$

6)

and the Riemann tensor is a map from three vector fields to a fourth:

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z \quad \text{— (19)}$$

where:

$$D_X = X^\mu D_\mu . \quad \text{— (20)}$$

In Cartan geometry these results are expressed elegantly in terms of the first and second Cartan structure equations:

$$T^a = D \wedge q^a \quad \text{— (21)}$$

$$R^a{}_b = D \wedge \omega^a{}_b \quad \text{— (22)}$$

The Jacobi identity is:

$$[[D_\lambda, D_\rho], D_\sigma] + [[D_\rho, D_\sigma], D_\lambda] + [[D_\sigma, D_\lambda], D_\rho] := 0 \quad \text{— (23)}$$

and this is important in field theory, because D_λ in group theory can be a group generator, such as a rotation generator, Lorentz boost generator, or space-time translation generator.