1)

Note 93(4): Confirmation of Correct Christoffel Symbols by Cartan

These are calculated as follows. The line element is:

$$ds^2 = -e^{2\alpha} dt^2 + e^{2\beta} dr^2 + r^2 d\Omega^2 \quad -(1)$$

So:

$$g_{00} = -e^{2\alpha},\ g_{11} = e^{2\beta},\ g_{22} = r^2,\ g_{33} = r^2 \sin^2\phi \quad -(2)$$

- The Christoffel symbol is:

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right) \quad -(3)$$

The inverse metrics are:

$$g^{00} = -e^{-2\alpha},\ g^{11} = e^{-2\beta},\ g^{22} = \frac{1}{r^2},\ g_{33} = \frac{1}{r^2 \sin^2\phi} \quad -(4)$$

- 1)

$$\Gamma^0_{00} = \frac{1}{2} g^{00} \left(\partial_0 g_{00} + \partial_0 g_{00} - \partial_0 g_{00} \right)$$

$$= \frac{1}{2} g^{00} \partial_0 g_{00} \quad -(5)$$

Here:

$$\partial_0 g_{00} = -\partial_0 \left(e^{2\alpha(r,t)} \right) \quad -(6)$$

$$= -2 (\partial_0 \alpha) e^{2\alpha}$$

$$\boxed{\Gamma^0_{00} = \partial_0 \alpha} \quad \checkmark\checkmark \quad -(7)$$

2)

$$\Gamma^0_{01} = \frac{1}{2} g^{0\rho}\left(\partial_0 g_{1\rho} + \partial_1 g_{\rho 0} - \partial_\rho g_{01}\right)$$

$$= \frac{1}{2} g^{00} \partial_1 g_{00} \quad \text{— (8)}$$

$$\boxed{\Gamma^0_{01} = \partial_1 \alpha} \quad \text{— (9)}$$

3)

$$\Gamma^1_{00} = \frac{1}{2} g^{1\rho}\left(\partial_0 g_{0\rho} + \partial_0 g_{\rho 0} - \partial_\rho g_{00}\right)$$

$$= -\frac{1}{2} g^{11} \partial_1 g_{00} \quad \text{— (10)}$$

$$\boxed{\Gamma^1_{00} = e^{2\alpha - 2\beta} \partial_1 \alpha} \quad \text{— (11)}$$

4)

$$\Gamma^0_{11} = \frac{1}{2} g^{0\rho}\left(\partial_1 g_{1\rho} + \partial_1 g_{\rho 1} - \partial_\rho g_{11}\right)$$

$$= -\frac{1}{2} g^{00} \partial_0 g_{11} \quad \text{— (12)}$$

$$\boxed{\Gamma^0_{11} = e^{2\beta - 2\alpha} \partial_0 \beta} \quad \text{— (13)}$$

5)

$$\Gamma^1_{01} = \frac{1}{2} g^{1\rho}\left(\partial_0 g_{1\rho} + \partial_1 g_{\rho 0} - \partial_\rho g_{01}\right)$$

$$= \frac{1}{2} g^{11} \partial_0 g_{11} \quad \text{— (14)}$$

$$\boxed{\Gamma^1_{01} = \partial_0 \beta} \quad \text{— (15)}$$

3) 6) $\Gamma^1{}_{11} = \frac{1}{2} g^{1\rho}\left(\partial_1 g_{1\rho} + \partial_1 g_{\rho 1} - \partial_\rho g_{11}\right)$

$= \frac{1}{2} g^{11} \partial_1 g_{11}$ — (16)

$\boxed{\Gamma^1{}_{11} = \partial_1 \beta}$ ✓✓ — (17)

7) $\Gamma^2{}_{12} = \frac{1}{2} g^{2\rho}\left(\partial_1 g_{2\rho} + \partial_2 g_{\rho 1} - \partial_\rho g_{12}\right)$

$= \frac{1}{2} g^{22} \partial_1 g_{22}$ — (18)

$g^{22} = \frac{1}{r^2}, \quad \partial_1 g_{22} = \frac{\partial}{\partial r}\left(r^2\right) = 2r$ — (19)

$\boxed{\Gamma^2{}_{12} = \frac{1}{r}}$ ✓✓ — (20)

8) $\Gamma^3{}_{13} = \frac{1}{2} g^{3\rho}\left(\partial_1 g_{3\rho} + \partial_3 g_{\rho 1} - \partial_\rho g_{13}\right)$

$= \frac{1}{2} g^{33} \partial_1 g_{33}$ — (21)

$g_{33} = r^2 \sin^2\phi, \quad g^{33} = \frac{1}{r^2 \sin^2\phi}$ — (22)

$\partial_1 g_{33} = 2r \sin^2\phi$

$\boxed{\Gamma^3{}_{13} = \frac{1}{r}}$ ✓✓ — (23)