1) 93(1): Calculation of Coulomb Law

This is:

$$\left(\underline{\nabla} \cdot \underline{E}\right)^0 = -\phi(0)\left(R^0{}_{101} + R^0{}_{202} + R^0{}_{303}\right) \quad -(1)$$

where:

$$R^0{}_{303} = R^0{}_{202} \sin^2\theta = \frac{\sin^2\theta}{r^2}\left(\frac{GM}{rc^2}\right)\left(1 - \frac{2GM}{rc^2}\right)^{-1} \quad -(2)$$

so:

$$R^0{}_{202} + R^0{}_{303} = \frac{1}{r^2}\left(1 + \sin^2\theta\right)\left(\frac{GM}{rc^2}\right)\left(1 - \frac{2GM}{rc^2}\right)^{-1} \quad -(3)$$

The $R^0{}_{101}$ Element

This is:

$$R^0{}_{101} = e^{2(\beta - \alpha)}\left(\partial_0(\partial_0\beta) + (\partial_0\alpha - \partial_0\beta)\partial_0\beta\right) - \partial_1(\partial_1\alpha) + (\partial_1\beta - \partial_1\alpha)\partial_1\alpha \quad -(4)$$

However:

$$\partial_0\beta = -\partial_0\alpha = 0 \quad -(5)$$

and

$$\alpha = -\beta \quad -(6)$$

so:

$$R^0{}_{101} = -2(\partial_1\alpha)^2 - \partial_1(\partial_1\alpha). \quad -(7)$$

Here:

$$2r\partial_1\alpha + 1 = e^{-2\alpha} \quad -(8)$$

$$e^{-2\alpha} = \left(1 - \frac{2GM}{rc^2}\right)^{-1} \quad -(9)$$

e

2)

So:
$$\partial_1 \alpha = \frac{1}{2r}\left(e^{-2\alpha} - 1\right)$$

$$R^0{}_{101} = -\frac{1}{2r^2}\left(e^{-2\alpha}-1\right)^2 - \frac{\partial}{\partial r}\left(\frac{1}{2r}\left(e^{-2\alpha}-1\right)\right)$$

$$= -\frac{1}{2r^2}\left(e^{-2\alpha}-1\right)^2 + \frac{1}{2r^2}\left(e^{-2\alpha}-1\right) - \frac{1}{2r}\frac{\partial}{\partial r}\left(e^{-2\alpha}-1\right)$$

$$= -\frac{1}{2r^2}\left(e^{-2\alpha}-1\right)^2 + \frac{1}{2r^2}\left(e^{-2\alpha}-1\right) + \frac{1}{r}\left(\partial_1\alpha\right)e^{-2\alpha}$$

$$= -\frac{1}{2r^2}\left(e^{-2\alpha}-1\right)^2 + \frac{1}{2r^2}\left(e^{-2\alpha}-1\right) + \frac{1}{2r^2}\left(e^{-2\alpha}-1\right)e^{-2\alpha}$$

$$= \frac{1}{2r^2}\left(e^{-2\alpha}-1\right)\left(1 - e^{-2\alpha} + 1 + e^{-2\alpha}\right)$$

$$= \frac{2}{2r^2}e^{-2\alpha} = \frac{1}{r^2}\left(1-\frac{2GM}{rc^2}\right)^{-1}$$

$$\boxed{R^0{}_{101} = \frac{1}{r^2}\left(1-\frac{2GM}{rc^2}\right)^{-1}} \quad \text{— (10)}$$

The Coulomb Law is therefore:

3)
$$\left(\underline{\nabla} \cdot \underline{E}\right)^{0} = -\phi^{(0)}\left(R^{0}{}_{101} + R^{0}{}_{202} + R^{0}{}_{303}\right)$$

$$= -\frac{\phi^{(0)}}{r^2}\left(1-\frac{2GM}{rc^2}\right)^{-1}\left(\left(1+\sin^2\theta\right)\frac{GM}{rc^2} + 1\right)$$

$$\rightarrow \frac{\phi^{(0)}}{r^2} \quad \text{as } rc^2 \gg 2GM. \qquad -(11)$$

Conclusion

$$\boxed{\left(\underline{\nabla} \cdot \underline{E}\right)^{0} = -\frac{\phi^{(0)}}{r^2}\left(1-\frac{2GM}{rc^2}\right)^{-1}\left(1 + \frac{MG}{rc^2}\left(1+\sin^2\theta\right)\right)}$$

$$-(12)$$

Notes

1) This is expressed in spherical polar coordinates.
2) There are various relativistic effects, for example the Coulomb law becomes a function of θ, indicating a rotation of polarization
3) In the lab.,
$$2GM \ll rc^2 \quad -(13)$$
so:
$$\left(\underline{\nabla} \cdot \underline{E}\right)^{0} = -\frac{\phi^{(0)}}{r^2} \quad -(14)$$
which is the usual Coulomb law.