APPENDIX 2 : EQUIVALENCE OF INDICES IN THE FIELD EQUATIONS

The homogeneous and inhomogeneous field equations can be written in equivalent ways, and the equivalence is proven in this Appendix. The first method of writing the homogeneous field equation is the sum:

$$\partial_\mu F^a_{\nu\rho} + \partial_\rho F^a_{\mu\nu} + \partial_\nu F^a_{\rho\mu} = \mu_0 \left(j^a_{\mu\nu\rho} + j^a_{\rho\mu\nu} + j^a_{\nu\rho\mu} \right) \quad -(B1)$$

where the charge current density three-forms are defined by:

$$j^a_{\mu\nu\rho} + j^a_{\rho\mu\nu} + j^a_{\nu\rho\mu} := -\frac{A^{(0)}}{\mu_0} \left(R^a_{\mu\nu\rho} + R^a_{\rho\mu\nu} + R^a_{\nu\rho\mu} + \omega^a_{\mu b} T^b_{\nu\rho} + \omega^a_{\rho b} T^b_{\mu\nu} + \omega^a_{\nu b} T^b_{\rho\mu} \right). \quad -(B2)$$

Consider individual tensor elements such as those defined by

$$\begin{aligned}
\partial_0 \tilde{F}^{a01} + \partial_2 \tilde{F}^{a21} + \partial_3 \tilde{F}^{a31} &= \frac{1}{2} |g| \bar{\epsilon}^{\mu 1 \rho\sigma} \partial_\mu F^a_{\rho\sigma} \\
&= \frac{1}{2} |g|^{1/2} \left(\bar{\epsilon}^{01\rho\sigma} \partial_0 F^a_{\rho\sigma} + \bar{\epsilon}^{21\rho\sigma} \partial_2 F^a_{\rho\sigma} + \bar{\epsilon}^{31\rho\sigma} \partial_3 F^a_{\rho\sigma} \right) \\
&= |g|^{1/2} \left(\partial_0 F^a_{23} + \partial_2 F^a_{30} + \partial_3 F^a_{02} \right) \quad -(B3)
\end{aligned}$$

which is a special case of the general result:

$$\partial_\mu \tilde{F}^{a\mu\nu} = |g|^{1/2} \left(\partial_\mu F^a_{\nu\rho} + \partial_\rho F^a_{\mu\nu} + \partial_\nu F^a_{\rho\mu} \right). \quad -(B4)$$

Now consider the following current term for $\sigma = 1$ to obtain:

$$\tilde{j}^{a\sigma} = \frac{1}{6} |g|^{1/2} \bar{\epsilon}^{\mu\nu\rho\sigma} j^a_{\mu\nu\rho}, \quad (\sigma = 1) \quad -(B5)$$

$$\tilde{j}^{a1} = \frac{1}{3} |g|^{1/2} \left(j^a_{023} + j^a_{302} + j^a_{230} \right). \quad -(B6)$$

Similarly, the other two current terms

$$\tilde{j}^{a\sigma} = \frac{1}{6}|g|^{1/2}\bar{\epsilon}^{\rho\mu\nu\sigma}\, \dot{j}^{a}_{\rho\mu\nu} \quad -(B7)$$

and

$$\tilde{j}^{a\sigma} = \frac{1}{6}|g|^{1/2}\bar{\epsilon}^{\nu\rho\mu\sigma}\, \dot{j}^{a}_{\nu\rho\mu} \quad -(B8)$$

give Eq. (B6) two more times. So the right hand side of Eq. (B1) for ν = 1 is:

$$\tilde{j}^{a1} = |g|^{1/2}\left(\dot{j}^{a}_{023} + \dot{j}^{a}_{302} + \dot{j}^{a}_{230}\right). \quad -(B9)$$

Finally use Eq. (A5) to find that:

$$\partial_\mu\left(|g|^{1/2}F^{a}_{\nu\rho}\right) = |g|^{1/2}\partial_\mu F^{a}_{\nu\rho} \quad -(B10)$$

and so derive Eq. (8) from Eq. (B1), Q.E.D. Note that the pre-multiplier $|g|^{1/2}$ cancels out on either side of Eq. (8).

Similarly it can be shown that the following expression of the inhomogeneous field equation:

$$\partial_\mu \tilde{F}^{a}_{\nu\rho} + \partial_\rho \tilde{F}^{a}_{\mu\nu} + \partial_\nu \tilde{F}^{a}_{\rho\mu} \quad -(B11)$$
$$= -A^{(0)}\left(\tilde{R}^{a}_{\mu\nu\rho} + \tilde{R}^{a}_{\rho\mu\nu} + \tilde{R}^{a}_{\nu\rho\mu} + \omega^{a}_{\mu b}\tilde{T}_{\nu\rho} + \omega^{a}_{\rho b}\tilde{T}_{\mu\nu} + \omega^{a}_{\nu b}\tilde{T}_{\rho\mu}\right)$$

is equivalent to:

$$\partial_\mu F^{a\mu\nu} = \mu_0 J^{a\nu} \quad -(B12)$$

as used in the text.

As a familiar example of Appendices 1 and 2 consider the Maxwell Heaviside (MH) equations in free space. The homogeneous MH equation in differential form notation is

$$d \wedge F = 0 \qquad \text{(B13)}$$

which is either:

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \qquad \text{(B14)}$$

or

$$\partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} = 0 \qquad \text{(B15)}$$

in tensor notation. The inhomogeneous MH equation in differential form notation is:

$$d \wedge \tilde{F} = 0 \qquad \text{(B16)}$$

which is either:

$$\partial_\mu F^{\mu\nu} = 0 \qquad \text{(B17)}$$

or

$$\partial_\mu \tilde{F}_{\nu\rho} + \partial_\rho \tilde{F}_{\mu\nu} + \partial_\nu \tilde{F}_{\rho\mu} = 0 \qquad \text{(B18)}$$

in tensor notation. The individual Hodge dual tensors are defined by:

$$\tilde{F}^{\nu\rho} = \frac{1}{2} \epsilon^{\nu\rho\mu\sigma} F_{\mu\sigma} \quad \text{etc.} \qquad \text{(B19)}$$

and indices are lowered as follows:

$$\tilde{F}_{\nu\rho} = g_{\nu\rho}\, g_{\rho\kappa}\, \tilde{F}^{\rho\kappa} \quad \text{etc.} \qquad \text{(B20)}$$

where $g_{\mu\nu}$ is the Minkowski metric in this case. The equivalent ECE equations in free space have the same properties exactly except of the addition of the index a to every tensor in the equations. Finally the homogeneous ECE equation in form notation is:

$$d \wedge F^a = \mu_0 j^a \quad \text{— (B21)}$$

which is

$$\partial_\mu \tilde{F}^{\mu\nu a} = \mu_0 \tilde{j}^{\nu a} \quad \text{— (B22)}$$

in tensor notation. The inhomogeneous ECE equation in form notation is:

$$d \wedge \tilde{F}^a = \mu_0 J^a \quad \text{— (B23)}$$

which is

$$\partial_\mu F^{\mu\nu a} = \mu_0 J^{\nu a} \quad \text{— (B24)}$$

in tensor notation. The individual Hodge duals are:

$$\tilde{F}^{\nu\rho a} = \frac{1}{2} |g|^{1/2} \bar{\epsilon}^{\nu\rho\mu\sigma} F^a_{\mu\sigma} \quad \text{etc.} \quad \text{— (B25)}$$

and indices are lowered with the metric of the base manifold:

$$\tilde{F}^a_{\nu\rho} = g_{\nu\rho}\, g_{\rho\kappa}\, \tilde{F}^{\nu\rho a} \quad \text{etc.} \quad \text{— (B26)}$$