The $R^0{}_{101}$ Element by Carroll

$$R^0{}_{101} = e^{2(\beta-\alpha)}\left(\partial_0^2\beta + (\partial_0\beta)^2 - \partial_0\alpha\,\partial_0\beta\right) + \partial_1\alpha\,\partial_1\beta - \partial_1^2\alpha - (\partial_1$$

$$- (1) \quad ((7.15))$$

Use:

$$\partial_0\alpha = \partial_0\beta = 0 \quad - (2) \quad ((7.17))$$

$$\alpha = -\beta \quad - (3) \quad ((7.22))$$

$$R^0{}_{101} = \partial_1\alpha\,\partial_1\beta - \partial_1^2\alpha - (\partial_1\alpha)^2 \quad - (4)$$

$$= -(\partial_1\alpha)^2 - (\partial_1\alpha)^2 - \partial_1(\partial_1\alpha) \quad - (5)$$

$$\boxed{R^0{}_{101} = -2(\partial_1\alpha)^2 - \partial_1(\partial_1\alpha)} \quad - (6)$$

Now use Carroll (7.23):

$$e^{2\alpha}(2r\,\partial_1\alpha + 1) = 1 \quad - (7) \quad ((7.23))$$

$$2r\,\partial_1\alpha + 1 = e^{-2\alpha}$$

$$\partial_1\alpha = \frac{1}{2r}\left(e^{-2\alpha} - 1\right) \quad - (8)$$

Now use Carroll (7.25):

$$e^{2\alpha} = 1 + \frac{\mu}{r} = 1 - x \quad - (9) \quad ((7.25))$$

So:

$$\partial_1\alpha = \frac{1}{2r}\left(\frac{1}{1-x} - 1\right)$$

$$\boxed{\begin{aligned} \partial_1\alpha &= \frac{1}{2r}\left(\frac{x}{1-x}\right) = -\partial_1\beta \\ e^{2\alpha} &= 1 - x \end{aligned}} \quad - (10)$$

So:

$$R^0{}_{101} = -2(\partial_1\alpha)^2 - \partial_1(\partial_1\alpha)$$

$$(\partial_1 \alpha)^2 = \frac{1}{4r^2}\left(\frac{x}{1-x}\right)^2 \quad \text{— (11)}$$

$$\partial_1(\partial_1 \alpha) = \frac{\partial}{\partial r}(\partial_1 \alpha) = \frac{1}{2}\left(\frac{x}{1-x}\right)\frac{\partial}{\partial r}\frac{1}{r} \quad \text{— (12}$$

$$= -\frac{1}{2r^2}\left(\frac{x}{1-x}\right)$$

$$R^0{}_{101} = -\frac{1}{2r^2}\left(\frac{x}{1-x}\right)^2 + \frac{1}{2r^2}\left(\frac{x}{1-x}\right) \quad \text{— (13)}$$

$$= -\frac{1}{2r^2}\left(\frac{x}{1-x}\right)\left(\frac{x}{1-x} - 1\right) \quad \text{— (14)}$$

✓ $$\boxed{R^0{}_{101} = -\frac{1}{2r^2}\left(\frac{x}{1-x}\right)\left(\frac{x}{1-x} - 1\right)} \quad \text{— (15)}$$

$$R^0{}_1{}^{01} = g^{00} g^{11} R^0{}_{101} \quad \text{— (16)}$$

$$R^0{}_1{}^{10} = -R^0{}_1{}^{01} = -g^{00} g^{11} R^0{}_{101} \quad \text{— (17)}$$

$$g_{00} = -(1-x), \quad g_{11} = \frac{1}{1-x} \quad \text{— (18)}$$

This is from Carroll (7.29):

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega^2 \quad \text{— (19)}$$

So: $$g^{00} = -\frac{1}{1-x}, \quad g^{11} = 1-x$$

$$-g^{00} g^{11} = 1$$ ✓

$$\boxed{R^0{}_1{}^{10} = R^0{}_{101}} \quad \text{— (20)}$$