# Triple Cross Check on Paper 88

We start from Carroll p. 19 eq (1.68):

$$T_{[\mu_1 \mu_2 \cdots \mu_n]} = \frac{1}{n!}\left(T_{\mu_1 \mu_2 \cdots \mu_n} + \text{antisymmetric sum over permutations of indices } \mu_1 \cdots \mu_n\right) \quad -(1)$$

Permutations which are the result of an odd number of exchanges are given a minus sign.

So:

$$\boxed{T_{[\mu\nu]} = \frac{1}{2}\left(T_{\mu\nu} - T_{\nu\mu}\right)} \quad -(2)$$

$$T_{[\mu\nu\rho]} = \frac{1}{6}\left(T_{\mu\nu\rho} - T_{\mu\rho\nu} + T_{\rho\mu\nu} - T_{\nu\mu\rho} + T_{\nu\rho\mu} - T_{\rho\nu\mu}\right) \quad -(3)$$

If:

$$T_{\mu\nu\rho} = -T_{\mu\rho\nu} \text{ etc.} \quad -(4)$$

$$\boxed{T_{[\mu\nu\rho]} = \frac{1}{3}\left(T_{\mu\nu\rho} + T_{\rho\mu\nu} + T_{\nu\rho\mu}\right)} \quad -(5)$$

The general rule is $n!\, m! / (n-m)!$ permutations, i.e.

1) When there are four indices the permutation rule is the same as for the $\epsilon_{0123}$ symbol:

$$
\left.\begin{array}{l}
0123 = 0312 = 0231 = 1 \\
1203 = 1320 = 1032 = 1 \\
2301 = 2013 = 2130 = 1 \\
3102 = 3210 = 3021 = 1 \\
0321 = 0213 = 0132 = -1 \\
1302 = 1023 = 1230 = -1 \\
2103 = 2310 = 2031 = -1 \\
3201 = 3012 = 3120 = -1
\end{array}\right] \quad -(6)
$$

So with:

$$\sigma = 0,\ \mu = 1,\ \nu = 2,\ \rho = 3 \quad -(7)$$

$$
\begin{aligned}
T_{[\sigma\mu\nu\rho]} = \frac{1}{12} \Big( & T_{\sigma\mu\nu\rho} + T_{\sigma\rho\mu\nu} + T_{\sigma\nu\rho\mu} \\
& + T_{\mu\nu\sigma\rho} + T_{\mu\rho\nu\sigma} + T_{\mu\sigma\rho\nu} \quad -(8) \\
& + T_{\nu\rho\sigma\mu} + T_{\nu\sigma\mu\rho} + T_{\nu\mu\rho\sigma} \\
& + T_{\rho\sigma\mu\nu} + T_{\rho\nu\mu\sigma} + T_{\rho\sigma\nu\mu} \Big)
\end{aligned}
$$

3) When we account for antisymmetry in the last two indices.

So :

$$D\wedge(R\wedge q) = \frac{1}{3} D\wedge R \qquad -(6)$$

$$= \frac{1}{12}\Big( D_\sigma R^a_{\mu\nu\rho} + D_\sigma R^a_{\rho\mu\nu} + D_\sigma R^a_{\nu\rho\mu}$$
$$+ D_\mu R^a_{\nu\sigma\rho} + D_\mu R^a_{\rho\nu\sigma} + D_\mu R^a_{\sigma\rho\nu}$$
$$+ D_\nu R^a_{\rho\sigma\mu} + D_\nu R^a_{\sigma\mu\rho} + D_\nu R^a_{\mu\rho\sigma}$$
$$+ D_\rho R^a_{\sigma\mu\nu} + D_\rho R^a_{\nu\mu\sigma} + D_\rho R^a_{\sigma\nu\mu}\Big).$$

and :

$$D\wedge(D\wedge T) = \frac{1}{3} D(D\wedge T)$$
$$= \frac{1}{12}\Big( D_\sigma D_\mu T^a_{\nu\rho} + D_\sigma D_\rho T^a_{\mu\nu} + D_\sigma D_\nu T^a_{\rho\mu}$$
$$+ D_\mu D_\nu T^a_{\sigma\rho} + D_\mu D_\rho T^a_{\nu\sigma} + D_\mu D_\sigma T^a_{\rho\nu}$$
$$+ D_\nu D_\rho T^a_{\sigma\mu} + D_\nu D_\sigma T^a_{\mu\rho} + D_\nu D_\mu T^a_{\rho\sigma}$$
$$+ D_\rho D_\sigma T^a_{\mu\nu} + D_\rho D_\nu T^a_{\mu\sigma} + D_\rho D_\sigma T^a_{\nu\mu}\Big)$$

$$-(7)$$

4)

The overall result is that if:

$$\boxed{\begin{aligned} D \wedge T &:= R \wedge q \quad &(8) \\ \text{then:} \quad D \wedge R &:= D(D \wedge T) \quad &(9) \end{aligned}}$$

<u>This is the true Bianchi identity, Q.E.D.</u>

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<u>Note</u>, <u>Number of Terms</u>

This is $N = \dfrac{m!\,n!}{(m-n)!}$.

a) When $m = 2$, $n = 1$, $N = 2$ (permute one from two indices)
b) When $m = 3$, $n = 1$, $N = 6$ (permute one from three indices)
c) When $m = 4$, $n = 2$, $N = 24$ (permute two from four indices)