THE BIANCHI IDENTITY OF DIFFERENTIAL GEOMETRY

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ABSTRACT

It is shown that the second Bianchi equation used by Einstein and Hilbert is incomplete, so that cosmology based on that equation is also incomplete. A great deal of new information can be obtained by deriving the true second Bianchi identity of differential geometry from first Bianchi identity of Cartan. When this done, it is seen that cosmology based on the Einstein Hilbert field equation is a narrow special case in which the torsion is missing. Using the true Bianchi identity, cosmology can be developed entirely in terms of torsion, in a simpler way, and providing more information.

Keywords : Second Bianchi identity of differential geometry, torsion based cosmology, ECE theory.

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1. INTRODUCTION

Recently, a generally covariant unified field theory has been developed {1-12} directly on standard differential geometry {13} using the Cartan structure equations and Bianchi equations {14}. This theory has been developed in order to suggest a logical geometrical framework for a unified field theory of natural philosophy. This is of course the philosophy upon which relativity theory is based, and asserts that geometry is the basis of natural philosophy. The theory is known as Einstein Cartan Evans (ECE) field theory because it aims to complete the well known work of Einstein and Cartan. In Section 2 it is shown that a new identity of differential geometry can be derived from the first Bianchi identity of differential geometry given by Cartan. It is shown that the true first and second Bianchi identities are related, and that both relate curvature to torsion. The traditional second Bianchi equation as used by Einstein and Hilbert {15} to derive the famous field equation is a narrow special case of the true Bianchi identity. So contemporary cosmology and relativity is also a narrow special case of what is possible. This conclusion is illustrated by ECE theory, in which the fundamental fields of physics are unified by the use of torsion to represent the electromagnetic field. It has also been shown that torsion is important {1-12} in a purely gravitational context, and is responsible for example for the formation of spiral galaxies.

In Section 3, a new field equation for cosmology is suggested using the novel concept of Noether forms to represent canonical energy momentum density. In contrast to the traditional Noether tensor of the Einstein Hilbert equation, a symmetric tensor, the Noether forms are anti-symmetric in their last two indices and are made proportional to the torsion form. This procedure greatly simplifies and also extends the field equation of Einstein and Hilbert.

2. DERIVATION OF THE TRUE SECOND BIANCHI IDENTITY

The first Bianchi identity as given by Cartan {13} is:

$$D \wedge T^a = d \wedge T^a + \omega^a{}_b \wedge T^b := R^a{}_b \wedge q^b \quad -(1)$$

where, in conventional notation, T^a is the torsion form, $\omega^a{}_b$ is the spin connection, $R^a{}_b$ is the curvature or Riemann form, and q^b is the tetrad form. In tensor notation {1-13}, Eq. (1) becomes:

$$\begin{aligned}
&\partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\rho} \\
&+ \partial_\nu \Gamma^\lambda_{\rho\mu} - \partial_\rho \Gamma^\lambda_{\nu\mu} + \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\rho\mu} - \Gamma^\lambda_{\rho\sigma}\Gamma^\sigma_{\nu\mu} \\
&+ \partial_\rho \Gamma^\lambda_{\mu\nu} - \partial_\mu \Gamma^\lambda_{\rho\nu} + \Gamma^\lambda_{\rho\sigma}\Gamma^\sigma_{\mu\nu} - \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\rho\nu} \\
&:= R^\lambda{}_{\rho\mu\nu} + R^\lambda{}_{\mu\nu\rho} + R^\lambda{}_{\nu\rho\mu}
\end{aligned} \quad -(2)$$

where $\Gamma^\lambda_{\nu\rho}$ is the gamma connection, and $R^\lambda{}_{\rho\mu\nu}$ is the Riemann tensor. Eq. (2) shows that the first Bianchi identity of Cartan is a true identity, it identifies the cyclic sum of three Riemann tensors to a cyclic sum of three definitions of the Riemann tensor. So the right hand side is identically equal to the left hand side, as required of a true identity. Note carefully that Eq. (2) is true for any kind of gamma connection, not just the Christoffel connection. The differential form notation is much more concise and elegant than the tensor notation, but both contain the same information. Implied in both equations (1) and (2) is a non-zero torsion tensor {1-13}:

$$T^\kappa{}_{\mu\nu} = \Gamma^\kappa_{\mu\nu} - \Gamma^\kappa_{\nu\mu} \neq 0. \quad -(3)$$

This vanishes for the symmetric Christoffel connection:

$$\Gamma^{\kappa}_{\mu\nu} = \Gamma^{\kappa}_{\nu\mu} \, . \quad -(4)$$

The first Bianchi equation used by Einstein and Hilbert, and in conventional cosmology, is

$$R^{a}{}_{b} \wedge q^{b} = 0 \quad -(5)$$

and this is not an identity because it is true if and only if the metric is symmetric, i.e. for a Christoffel connection. Eq. (5) is often mistakenly known as "the first Bianchi identity", but it is not a true identity. It was actually discovered {13} by Ricci and Levi-Civita, and not by Bianchi. In tensor notation, Eq. (5) is:

$$R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0 \quad -(6)$$

where:

$$R_{\rho\sigma\mu\nu} = g_{\rho\kappa} R^{\kappa}{}_{\sigma\mu\nu} \quad -(7)$$

is the Riemann tensor with indices lowered. Here

$$g_{\rho\kappa} = g_{\kappa\rho} \quad -(8)$$

is the symmetric metric. Note that Eqs (5) and (6) imply a vanishing torsion:

$$T^{\kappa}_{\mu\nu} = \Gamma^{\kappa}_{\mu\nu} - \Gamma^{\kappa}_{\nu\mu} = 0. \quad -(9)$$

In Riemann normal coordinates {13} Eq. (6) is:

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\gamma\beta\delta}$$

$$= \frac{1}{2}\Big(\partial_\beta \partial_\gamma g_{\alpha\delta} - \partial_\alpha \partial_\gamma g_{\beta\delta} - \partial_\beta \partial_\delta g_{\alpha\gamma} + \partial_\alpha \partial_\gamma g_{\beta\gamma}$$

$$+ \partial_\gamma \partial_\delta g_{\alpha\beta} - \partial_\alpha \partial_\delta g_{\gamma\beta} - \partial_\gamma \partial_\beta g_{\alpha\delta} + \partial_\alpha \partial_\beta g_{\gamma\delta}$$

$$+ \partial_\delta \partial_\beta g_{\alpha\gamma} - \partial_\alpha \partial_\beta g_{\alpha\gamma} - \partial_\delta \partial_\gamma g_{\alpha\beta} + \partial_\alpha \partial_\gamma g_{\delta\beta}\Big)$$

$$= 0 \qquad -(10)$$

and it is seen that this is not true of the metric is not symmetric, and is not true if the connection is not a Christoffel connection. For the general connection the true first Bianchi identity is Eq (1) or Eq. (2) and Eq. (1) is the field equation of classical electrodynamics in ECE theory {1-12}.

A second Bianchi equation of Cartan geometry is given in ref. {13} as:

$$D \wedge R^a{}_b = 0 \qquad -(11)$$

but this is a special case {13} when the torsion is zero (the Einstein Hilbert case). In tensor notation Eq. (11) is:

$$D_\lambda R_{\rho\sigma\mu\nu} + D_\rho R_{\sigma\lambda\mu\nu} + D_\sigma R_{\lambda\rho\mu\nu} = 0 \qquad -(12)$$

and is always referred to as "the second Bianchi identity" in conventional cosmology. In fact it is not an identity. This fact is shown {13} by expanding it in Riemann normal coordinates:

$$
\begin{aligned}
& D_\lambda R_{\rho\sigma\mu\nu} + D_\rho R_{\sigma\lambda\mu\nu} + D_\sigma R_{\lambda\rho\mu\nu} \\
&= \frac{1}{2}\Big(\boxed{\partial_\lambda \partial_\mu \partial_\sigma g_{\rho\nu}} - \partial_\lambda \partial_\mu \partial_\rho g_{\nu\sigma} - \partial_\lambda \partial_\nu \partial_\sigma g_{\rho\mu} + \partial_\lambda \partial_\nu \partial_\rho g_{\mu\sigma} \\
&\quad + \partial_\rho \partial_\mu \partial_\lambda g_{\sigma\nu} - \partial_\rho \partial_\mu \partial_\sigma g_{\nu\lambda} - \partial_\rho \partial_\nu \partial_\lambda g_{\sigma\mu} + \partial_\rho \partial_\nu \partial_\sigma g_{\mu\lambda} \\
&\quad + \partial_\sigma \partial_\mu \partial_\rho g_{\lambda\nu} - \boxed{\partial_\sigma \partial_\mu \partial_\lambda g_{\nu\rho}} - \partial_\sigma \partial_\nu \partial_\rho g_{\lambda\mu} + \partial_\sigma \partial_\nu \partial_\lambda g_{\mu\rho} \Big) \\
&= 0. \qquad -(13)
\end{aligned}
$$

It is seen by comparison of the dotted terms in the above equation that a zero result is obtained if and only if:

$$g_{\rho\nu} = g_{\nu\rho} \qquad -(14)$$

where the commutation of partial four-derivatives has been used:

$$\partial_\lambda \partial_\mu \partial_\sigma = \partial_\mu \partial_\sigma \partial_\lambda , \qquad -(15)$$

$$\partial_\mu \partial_\sigma \partial_\lambda = \partial_\sigma \partial_\mu \partial_\lambda . \qquad -(16)$$

Eq. (14) is true if and only if the connection is the Christoffel connection:

$$\Gamma^{\kappa}_{\mu\nu} = \Gamma^{\kappa}_{\nu\mu} \qquad -(17)$$

and if and only if the torsion tensor vanishes:

$$T^{\kappa}_{\mu\nu} = \Gamma^{\kappa}_{\mu\nu} - \Gamma^{\kappa}_{\nu\mu} = 0. \qquad -(18)$$

Eq. (12) can be rewritten using contraction of indices {13} as:

$$D^{\mu} G_{\nu\mu} = 0 \quad -(19)$$

where $G_{\nu\mu}$ is the Einstein tensor:

$$G_{\nu\mu} = R_{\nu\mu} - \frac{1}{2} R\, g_{\nu\mu}. \quad -(20)$$

Here $R_{\nu\mu}$ is the symmetric Ricci tensor {13} and R is the conventional scalar curvature of cosmology {13}. The Einstein Hilbert field equation is obtained by making Eq. (19) proportional to the Noether Theorem:

$$D^{\mu} T_{\nu\mu} = 0 \quad -(21)$$

i.e.

$$D^{\mu} G_{\nu\mu} = k\, D^{\mu} T_{\nu\mu} \quad -(22)$$

and using the solution:

$$G_{\nu\mu} = k\, T_{\nu\mu} \quad -(23)$$

which is the Einstein Hilbert field equation of 1915. Here:

$$T_{\nu\mu} = T_{\mu\nu} \quad -(24)$$

is the symmetric canonical energy momentum tensor of Noether, and Eq. (21) denotes conservation of energy - momentum as is well known.

It is seen that the famous field equation and the equally famous "second Bianchi identity" are true if and only if the connection is symmetric and if and only if the torsion tensor is zero. They are no longer true otherwise, so conventional cosmology is severely constrained by these assumptions {1-12}.

The true second Bianchi identity is obtained by taking the D ^ derivative of both sides of the true first Bianchi identity (1). Thus:

$$D \wedge (R^a{}_b \wedge q^b) := D \wedge (D \wedge T^a). \quad -(25)$$

The general rule for the exterior derivative of an n-form is {13}

$$(d \wedge A)_{\mu_1 \cdots \mu_{p+1}} = (p+1)\, \partial_{[\mu_1} A_{\mu_2 \cdots \mu_{p+1}]} . \quad -(26)$$

The derivatives in Eq. (25) need consideration of:

$$(d \wedge A)_{\mu_1 \mu_2 \mu_3 \mu_4} = 4\, \partial_{[\mu_1} A_{\mu_2 \mu_3 \mu_4]} \quad -(27)$$

where the four cyclic permutations are:

$$\mu_1 \mu_2 \mu_3 \mu_4 ,\ \mu_4 \mu_1 \mu_2 \mu_3 ,\ \mu_3 \mu_4 \mu_1 \mu_2 ,\ \mu_2 \mu_3 \mu_4 \mu_1 . \quad -(28)$$

We denote:

$$\mu_1 = \mu,\ \mu_2 = \nu,\ \mu_3 = \rho,\ \mu_4 = \sigma. \quad -(29)$$

In tensor notation:

$$D \wedge (R^a{}_b \wedge q^b) = D \wedge (R^a{}_{\mu\nu\rho} + R^a{}_{\nu\rho\mu} + R^a{}_{\rho\mu\nu}) \quad -(30)$$

so:

$$D \wedge (R^a{}_b \wedge q^b) = D \wedge R^a{}_{\mu\nu\rho} + D \wedge R^a{}_{\nu\rho\mu} + D \wedge R^a{}_{\rho\mu\nu} . \quad -(31)$$

This is a sum of twelve terms:

$$
\begin{aligned}
D\wedge\left(R^a{}_b\wedge q^b\right)
&= D_\sigma R^a{}_{\mu\nu\rho} + D_\mu R^a{}_{\nu\rho\sigma} + D_\rho R^a{}_{\sigma\mu\nu} + D_\nu R^a{}_{\rho\sigma\mu} \\
&+ D_\sigma R^a{}_{\nu\rho\mu} + D_\mu R^a{}_{\sigma\nu\rho} + D_\rho R^a{}_{\mu\sigma\nu} + D_\nu R^a{}_{\rho\mu\sigma} \\
&+ D_\sigma R^a{}_{\rho\mu\nu} + D_\nu R^a{}_{\sigma\rho\mu} + D_\mu R^a{}_{\nu\sigma\rho} + D_\rho R^a{}_{\mu\nu\sigma} .
\end{aligned}
\quad -(32)
$$

Now use the anti-symmetry properties {13}:

$$D_\nu R^a{}_{\rho\sigma\mu} = -D_\nu R^a{}_{\rho\mu\sigma}, \quad -(33)$$

$$D_\mu R^a{}_{\nu\rho\sigma} = -D_\mu R^a{}_{\nu\sigma\rho}, \quad -(34)$$

$$D_\rho R^a{}_{\mu\sigma\nu} = -D_\rho R^a{}_{\mu\nu\sigma}, \quad -(35)$$

to obtain:

$$
\begin{aligned}
D\wedge\left(R^a{}_b\wedge q^b\right)
&= D_\sigma\left(R^a{}_{\mu\nu\rho} + R^a{}_{\nu\rho\mu} + R^a{}_{\rho\mu\nu}\right) \\
&+ D_\rho R^a{}_{\sigma\mu\nu} + D_\mu R^a{}_{\sigma\nu\rho} + D_\nu R^a{}_{\sigma\rho\mu} .
\end{aligned}
\quad -(36)
$$

In condensed notation this is written as:

$$D\wedge\left(R^a{}_b\wedge q^b\right) = D\wedge R^a + D\left(R^a{}_b\wedge q^b\right). \quad -(37)$$

Similarly:

$$
\begin{aligned}
D\wedge\left(D\wedge T^a\right) &= D_\sigma\left(D_\mu T^a{}_{\nu\rho} + D_\nu T^a{}_{\rho\mu} + D_\rho T^a{}_{\mu\nu}\right) \\
&+ D_\rho D_\sigma T^a{}_{\mu\nu} + D_\mu D_\sigma T^a{}_{\nu\rho} + D_\nu D_\sigma T^a{}_{\rho\mu}
\end{aligned}
\quad -(38)
$$

which in condensed notation is:

$$D \wedge (D \wedge T^a) = D(D \wedge T^a) + D \wedge (D T^a). \quad -(39)$$

Comparing Eqs. (37) and (39) gives:

$$D \wedge R^a{}_b := q^\sigma_b \left(D_\rho D_\sigma T^a_{\mu\nu} + D_\mu D_\sigma T^a_{\nu\rho} + D_\nu D_\sigma T^a_{\rho\mu} \right) \quad -(40)$$

which in condensed notation can be written as:

$$D \wedge R^a{}_b := q^\sigma_b D \wedge (D_\sigma T^a). \quad -(41)$$

Eqs. (40) or (41) denote the true second Bianchi identity. So the complete set of

Cartan equations are:

$$T^a = D \wedge q^a, \quad R^a{}_b = D \wedge \omega^a{}_b,$$
$$D \wedge T^a := R^a{}_b \wedge q^b, \quad D \wedge R^a{}_b := q^\sigma_b D \wedge (D_\sigma T^a). \quad -(42)$$

The Einstein Hilbert field theory is the special case:

$$\left.\begin{array}{ll} T^a = 0, & R^a{}_b = D \wedge \omega^a{}_b, \\ R^a{}_b \wedge q^b = 0, & D \wedge R^a{}_b = 0, \end{array}\right\} \quad -(43)$$

and misses a great deal of basic information. It is seen that there is only on true Bianchi

identity because Eq. (41) has been derived from Eq. (1).