8(5): Details of the First and Second Bianchi Identities

a) Absence of Torsion

The "first Bianchi identity" in condensed notation is

$$R^a{}_b \wedge q^b = 0 \qquad (1)$$

and in Riemann notation is:

$$R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0 \qquad (2)$$

It is not, in fact, an identity and was discovered by Ricci, not Bianchi. It is true if and only if there is no torsion. It is the result of the following symmetries of the Riemann tensor:

$$R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu} \qquad (3)$$

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu} \qquad (4)$$

In Riemann normal coordinates (vol. 2, p. 198)

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\partial_\beta \partial_\gamma g_{\alpha\delta} - \partial_\alpha \partial_\gamma g_{\beta\delta} - \partial_\beta \partial_\delta g_{\alpha\gamma} + \partial_\alpha \partial_\delta g_{\beta\gamma}\right) \qquad (5)$$

so g in eq. (18.32) of vol. 2 of GCUFT,

eq. (2) follows from:

$$g_{\alpha\beta} = g_{\beta\alpha}. \qquad (6)$$

2) In the presence of torsion, Cartan showed that:

$$D \wedge T^a := R^a{}_b \wedge q^b \quad -(7)$$

and this is a true identity. Here T^a is the Cartan torsion form.

The "second Bianchi identity" is:

$$D \wedge R^a{}_b = 0 \quad -(8)$$

where:

$$R^a{}_b = D \wedge \omega^a{}_b . \quad -(9)$$

Again, this is true only in the absence of torsion. This is not made clear by Carroll. This omission of torsion omits much physics. Using the definition:

$$R_{\rho\sigma\mu\nu} = g_{\rho\lambda} R^\lambda{}_{\sigma\mu\nu} \quad -(10)$$

the "second Bianchi identity" can be written as

$$D_\lambda R_{\rho\sigma\mu\nu} + D_\rho R_{\sigma\lambda\mu\nu} + D_\sigma R_{\lambda\rho\mu\nu} = 0 \quad -(11)$$

As in Carroll, 1997 notes, page 80, we evaluate this in Riemann normal coordinates:

$$D_\lambda R_{\rho\sigma\mu\nu} = \partial_\lambda R_{\rho\sigma\mu\nu} \qquad \text{— (12)}$$

$$= \frac{1}{2}\partial_\lambda\left(\partial_\mu\partial_\sigma g_{\rho\nu} - \partial_\mu\partial_\rho g_{\nu\sigma} - \partial_\nu\partial_\sigma g_{\rho\mu} + \partial_\nu\partial_\rho g_{\mu\sigma}\right)$$

So:

$$D_\lambda R_{\rho\sigma\mu\nu} + D_\rho R_{\sigma\lambda\mu\nu} + D_\sigma R_{\lambda\rho\mu\nu}$$

$$= \frac{1}{2}\Big(\boxed{\partial_\lambda\partial_\mu\partial_\sigma g_{\rho\nu}} - \partial_\lambda\partial_\mu\partial_\rho g_{\nu\sigma} - \partial_\lambda\partial_\nu\partial_\sigma g_{\rho\mu}$$

$$+ \partial_\lambda\partial_\nu\partial_\rho g_{\mu\sigma} + \partial_\rho\partial_\mu\partial_\lambda g_{\sigma\nu} - \partial_\rho\partial_\mu\partial_\sigma g_{\nu\lambda}$$

$$- \partial_\rho\partial_\nu\partial_\lambda g_{\sigma\mu} + \partial_\rho\partial_\nu\partial_\sigma g_{\mu\lambda} + \partial_\sigma\partial_\mu\partial_\rho g_{\lambda\nu}$$

$$\boxed{- \partial_\sigma\partial_\mu\partial_\lambda g_{\nu\rho}} - \partial_\sigma\partial_\nu\partial_\rho g_{\lambda\mu} + \partial_\sigma\partial_\nu\partial_\lambda g_{\mu\rho}\Big)$$

$$= 0. \qquad \text{— (13)}$$

There are twelve terms and they cancel out to zero. For example, in the dotted terms:

$$g_{\rho\nu} = g_{\nu\rho} \qquad \text{— (14)}$$

and

$$\partial_\lambda\partial_\mu\partial_\sigma = \partial_\mu\partial_\lambda\partial_\sigma \qquad \text{— (15)}$$

$$\partial_\lambda\partial_\mu\partial_\sigma = \partial_\sigma\partial_\lambda\partial_\mu = \partial_\sigma\partial_\mu\partial_\lambda \qquad \text{— (16)}$$

4)

These results are true only for a symmetric connection and a symmetric metric.

Summary

In the absence of torsion:

$$\boxed{\begin{aligned} R^a{}_b \wedge q^b &= 0 \\ D \wedge R^a{}_b &= 0 \end{aligned}} \quad \text{--- (17)}$$

b) Presence of Torsion

$$\boxed{\begin{aligned} R^a{}_b \wedge q^b &:= D \wedge T^a \neq 0 \\ D \wedge R^a{}_b &\neq 0 \end{aligned}} \quad \text{--- (18)}$$

Where:

$$T^a = D \wedge q^a \quad \text{--- (19)}$$

$$R^a{}_b = D \wedge \omega^a{}_b \quad \text{--- (20)}$$

The Einstein field equation is based on

$$D \wedge R^a{}_b = 0 \quad \text{--- (21)}$$

so is changed in the presence of torsion.