58(4): Some Remarks on Conservation of Energy in General Relativity

The Noether Theorem is fundamental to the Einstein Hilbert laws of gravitational general relativity the Noether Theorem express conservation of energy – momentum as follows:

$$D^{\mu} T_{\mu\nu} = 0 \quad - (1)$$

where $T_{\mu\nu}$ is the canonical energy momentum tensor:

$$T_{\mu\nu} = T_{\nu\mu} \quad - (2)$$

Einstein derived his field equation of 1915 from the property of the Riemann tensor in the absence of torsion:

$$D_{[\lambda} R_{\rho\sigma]\mu\nu} = 0. \quad - (3)$$

Eq (3) can be rewritten as:

$$D^{\mu} G_{\mu\nu} = 0 \quad - (4)$$

where:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \quad - (5)$$

is the Einstein field tensor.

Einstein assumed that eq. (4) is proportional to eq. (1):

2)

$$D^\mu G_{\mu\nu} = k D^\mu T_{\mu\nu} \quad \text{— (6)}$$

i.e.

$$\boxed{G_{\mu\nu} = k T_{\mu\nu}} \quad \text{—— (7)}$$

which is the famous field equation.

It is seen that the Noether Theorem is made proportional to eq. (3), which is known as the second Bianchi identity. So conservation of energy and momentum (Noether Theorem) is built in to general relativity at a basic level.

However, in using eq. (3) Einstein did not consider the effect of torsion. As stated by Carroll (1997 notes, after eq. (3.88)) there are additional terms in the eq. (3) (or eq. (4)) if torsion is considered. This means that torsion introduces extra terms into the Noether Theorem itself. This means that in the presence of torsion, ~~the~~ Einstein field equation (7) must be extended.

If we consider the Riemann form $R^a{}_b$ in eq. (3):

3)

$$D_{[\lambda} R_{\rho\sigma]}{}^{a}{}_{b} = 0$$

i.e.

$$\boxed{D \wedge R^{a}{}_{b} = 0} \quad -(8)$$

and this is the second Bianchi identity as given by Carroll in chapter 3. This is true if and only if the connection is symmetric:

$$\Gamma^{\kappa}{}_{\mu\nu} = \Gamma^{\kappa}{}_{\nu\mu}. \quad -(9)$$

In the presence of torsion:

$$D \wedge R^{a}{}_{b} \neq 0 \quad -(10)$$

If we introduce the Noether form $T^{a}{}_{b}$, then the EH field equation is:

$$\boxed{D \wedge R^{a}{}_{b} = k D \wedge T^{a}{}_{b} = 0} \quad -(11)$$

and the Noether Theorem is:

$$D \wedge T^{a}{}_{b} = 0 \quad -(12)$$

However, if there is torsion:

$$T^{\kappa}{}_{\mu\nu} = \Gamma^{\kappa}{}_{\mu\nu} - \Gamma^{\kappa}{}_{\nu\mu} \neq 0 \quad -(13)$$

then the field equation is:

$$\boxed{D \wedge R^{a}{}_{b} = k D \wedge T^{a}{}_{b} \neq 0}$$

$-(14)$

4)

In both cases (eqs. (11) and (14)):

$$\boxed{R^a{}_b = kT^a{}_b} \quad -(15)$$

which is the ECE field equation that generalizes the EH field equation to consider both curvature and torsion. Energy momentum is conserved under all circumstances.

In addition to eq. (15) we have the Cartan equation:

$$D \wedge T^a = R^a{}_b \wedge q^b \quad -(16)$$

where:

$$T^a = D \wedge q^a \quad -(17)$$

$$R^a{}_b = D \wedge \omega^a{}_b \quad -(18)$$

In Einstein Hilbert theory:

$$R^a{}_b \wedge q^b = 0 \quad -(19)$$

$$T^a = 0. \quad -(20)$$

The field equations of ECE are obtained from:

$$A^a = A^{(0)} q^a \quad -(21)$$

$$\Longrightarrow F^a = A^{(0)} T^a \quad -(22)$$

and are, from eq. (16):

5)

$$d \wedge F^a = \mu_0 j^a \quad -(23)$$

$$d \wedge \tilde{F}^a = \mu_0 \tilde{j}^a \quad -(24)$$

where:

$$j^a = \frac{A^{(0)}}{\mu_0} \left(R^a{}_b \wedge q^b - \omega^a{}_b \wedge T^b \right) \quad -(25)$$

$$\tilde{j}^a = \frac{A^{(0)}}{\mu_0} \left(\tilde{R}^a{}_b \wedge q^b - \omega^a{}_b \wedge \tilde{T}^b \right) \quad -(26)$$

Here the tilde denotes Hodge dual in four dimensions.

<u>Pure Rotation</u>

In this case:

$$R^a{}_b \wedge q^b = \omega^a{}_b \wedge T^b. \quad -(27)$$

This means that there are no central forces present, and

$$R^a{}_b = -\frac{1}{2} \kappa \epsilon^a{}_{bc} T^c \quad -(28)$$

so $R^a{}_b$ is dual to T^c in the tangent spacetime at P to the base manifold. In this case $R^a{}_b$ and T^c both express rotational motion. We can write:

$$F^a{}_b = -\frac{1}{2} \kappa \epsilon^a{}_{bc} F^c \quad -(29)$$

6)

So the conservation law in this case is

$$\boxed{F^a{}_b = kA^{(0)} T^a{}_b} \quad -(30)$$

where $T^a{}_b$ is the antisymmetric part of the Noether form. Eq. (30) expresses conservation of angular energy, angular momentum. We have:

$$d \wedge F^a{}_b = d \wedge T^a{}_b \text{ (antisymmetric)}$$
$$= 0$$
$$= d \wedge F^c \quad -(31)$$

This means that there is no current j^c or $\tilde{j}^c$ and the field spins indefinitely.

So conservation of energy momentum is built in to the ECE theory in all manifestations