85(6): Lamb Shift in H Atom

The Dirac equation of the H atom is:

$$\left(i\gamma^\mu \partial_\mu - \frac{mc}{\hbar} - \frac{\alpha}{r} \right)\psi = 0 \quad -(1)$$

where

$$\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} \quad -(2)$$

is the fine structure constant. The solution of eq (1) is given by:

E. Merzbacher, "Quantum Mechanics" (Wiley, 1970, 2nd. ed.), eq. (24.83):

$$E = mc^2 \left(1 + \frac{\alpha^2}{\left(\left(\left(j+\frac{1}{2}\right)^2 - \alpha^2 \right)^{1/2} + n' \right)^2} \right)^{1/2} \quad -(3)$$

where

$$\left. \begin{array}{l} j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \; ; \; n' = 0, 1, 2 \\ n \text{ (non-relativistic)} = j + 1/2 + n' \end{array} \right\} -(4)$$

In the absence of the vacuum potential pairs of states with same j but opposite parity are degenerate. This is because $\lambda = j + 1/2$ appears squared in eq. (3). For H:

$2S_{1/2}$ is $n = 2, \ell = 0, \boxed{j = 1/2,}\ s = 1/2$

$2P_{1/2}$ is $n = 2, \ell = 1, \boxed{j = 1/2,}\ s = 1/2$

2) where:

$$j = l+s,\ l+s-1,\ \dots,\ |l-s| \quad -(5)$$

So from the Dirac equation (1) the energy levels of $2S_{1/2}$ and $2P_{1/2}$ are the same.

The Lamb shift shows that they are in fact different by 1060 MHz, the $2S_{1/2}$ state being of higher energy.

This is explained in ECE theory using the result:

$$\gamma^\mu \rightarrow \gamma^\mu \left(1 + \frac{\alpha}{4\pi}\right) \quad -(6)$$

which gives the correct g factor of the electron to one part in a million. So eq. (1) becomes:

$$\boxed{\left(i\gamma^\mu \left(1 + \frac{\alpha}{4\pi}\right)\partial_\mu - \frac{mc}{\hbar} - \frac{\alpha}{r}\right)\psi = 0 \quad -(7)}$$

The solution of this equation removes the degeneracy between $2S_{1/2}$ and $2P_{1/2}$ of the Dirac equation of H. This is the Lamb shift.

It is seen that eq (7) shifts r of the Coulomb potential by an amount:

$$-\frac{\alpha}{r}\psi \rightarrow -\frac{\alpha}{r}\psi + \frac{i\gamma^\mu \alpha}{4\pi}\partial_\mu \psi \quad -(8)$$

3) This contribution to r from the vacuum is given by

$$i\gamma^\mu \partial_\mu \psi = 4\pi r_{vac} \psi \quad -(9)$$

so the expectation value is:

$$\langle r_{vac} \rangle = \overline{\psi}\, r_{vac}\, \psi. \quad -(10)$$

The Dirac equation of the H atom in the presence of α from the vacuum is therefore:

$$\left(i\gamma^\mu \partial_\mu - \frac{mc}{\hbar} + \alpha\left(\frac{1}{\langle r_{vac} \rangle} - \frac{1}{r} \right)\right)\psi$$

$-(11)$

The Lamb shift is ~~therefore~~ therefore:

$$\boxed{\Delta E = \hbar \alpha c \left(\frac{1}{\langle r_{vac} \rangle_{2S_{1/2}}} - \frac{1}{\langle r_{vac} \rangle_{2P_{1/2}}} \right)}$$

$-(12)$

in joules, i.e.

$$\Delta \bar{\nu} = \alpha \left(\frac{1}{\langle r_{vac} \rangle_{2S_{1/2}}} - \frac{1}{\langle r_{vac} \rangle_{2P_{1/2}}} \right)$$

$-(13)$

in wavenumbers (cm^{-1}).

1 wavenumber = 30 GHz $\quad -(14)$

4) The mathematical problem is to solve the following equations simultaneously:

$$\left(i\gamma^\mu \partial_\mu - \frac{mc}{\hbar} + \alpha\left(\frac{1}{\langle r_{vac}\rangle} - \frac{1}{r}\right)\right)\psi = 0 \quad -(15)$$

$$i\gamma^\mu \partial_\mu \psi = 4\pi r_{vac}\psi \quad -(16)$$

In order to avoid singularities this should be done numerically, but not using the methods of path integral formalism. These are two equations in two unknowns, $\langle r_{vac}\rangle$ and ψ, where

$$\langle r_{vac}\rangle = \overline{\psi}\, r_{vac}\, \psi. \quad -(17)$$

In the first approximation, ψ is the Dirac spinor of the H atom from eqn. (1). Thus $\overline{\psi}$ is its adjoint. For $2S_{1/2}$ and $2P_{1/2}$, ψ and $\overline{\psi}$ are different, so $\langle r_{vac}\rangle$ is different, giving a first approximation to the Lamb shift.

This is the method used by Bethe, but he did not have available powerful computer methods to solve (15) and (16) simultaneously.