1)

74(3): Example of Tetrad from the Circular Complex Basis, Comparison with Quaternions

The complex circular basis is defined by:

$$\left.\begin{aligned} \underline{q}^{(1)} \times \underline{q}^{(2)} &= i\,\underline{q}^{(3)*} \\ \underline{q}^{(2)} \times \underline{q}^{(3)} &= i\,\underline{q}^{(1)*} \\ \underline{q}^{(3)} \times \underline{q}^{(1)} &= i\,\underline{q}^{(2)*} \end{aligned}\right] \quad -(1)$$

where $*$ denotes complex conjugate. It is related to the Cartesian basis by:

$$\left.\begin{aligned} \underline{q}^{(1)} &= \frac{1}{\sqrt{2}}(\underline{i} - i\underline{j}) \\ \underline{q}^{(2)} &= \frac{1}{\sqrt{2}}(\underline{i} + i\underline{j}) \\ \underline{q}^{(3)} &= \underline{k}. \end{aligned}\right] \quad -(2)$$

In three dimensions, the Cartan tetrad from this basis is:

$$q^a_\mu = \begin{bmatrix} q^{(1)}_X & q^{(1)}_Y & q^{(1)}_Z \\ q^{(2)}_X & q^{(2)}_Y & q^{(2)}_Z \\ q^{(3)}_X & q^{(3)}_Y & q^{(3)}_Z \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -(3)$$

The Cartan tetrad is always defined by

2)

$$V^a = g_\mu^{\ a} V^\mu \quad -(4)$$

If V^μ denotes the Cartesian components of a vector field whose basis elements are $\underline{i}$, $\underline{j}$ and $\underline{k}$ then:

$$V^1 = V_X = V^2 = V_Y = V^3 = V_Z = 1. \quad -(5)$$

So:

$$V^a = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad -(6)$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}}(1-i) \\ \frac{1}{\sqrt{2}}(1+i) \\ 1 \end{bmatrix}. \quad -(6a)$$

Thus:

$$\left. \begin{aligned} V^{(1)} &= \frac{1}{\sqrt{2}}(1-i), \\ V^{(2)} &= \frac{1}{\sqrt{2}}(1+i) \\ V^{(3)} &= 1. \end{aligned} \right\} \quad -(7)$$

Therefore there are two distinct 3-D spaces.

3) The tetrad is defined by the superposition of spaces in eq. (4). This is an extension of Cartan's analysis in which a was a tangent space and μ a base manifold.

<u>Four - Dimensional Spacetime</u>

In this case:

$$q_\mu^{\ a} = \left(q_0^{\ a}, -\underline{q}^{\ a} \right) \quad -(8)$$

If:

$$a = (0) \quad -(9)$$

is defined as time - like then its only possible index is also timelike:

$$\mu = 0 . \quad -(10)$$

So:

$$q_0^{(0)} = 1 , \quad -(11)$$

and:

$$q_X^{(0)} = q_Y^{(0)} = q_Z^{(0)} = 0 . \quad -(12)$$

Thus:

$$q_\mu^{\ a} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad -(13)$$

4) The 4-D tetrad is defined by:

$$\begin{bmatrix} V^{(0)} \\ V^{(1)} \\ V^{(2)} \\ V^{(3)} \end{bmatrix} = \begin{bmatrix} q^{(0)}_0 & q^{(0)}_1 & q^{(0)}_2 & q^{(0)}_3 \\ q^{(1)}_0 & q^{(1)}_1 & q^{(1)}_2 & q^{(1)}_3 \\ q^{(2)}_0 & q^{(2)}_1 & q^{(2)}_2 & q^{(2)}_3 \\ q^{(3)}_0 & q^{(3)}_1 & q^{(3)}_2 & q^{(3)}_3 \end{bmatrix} \begin{bmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -\frac{1}{\sqrt{2}}(1-i) \\ -\frac{1}{\sqrt{2}}(1+i) \\ -1 \end{bmatrix} \quad -(14)$$

Normalization Condition and Quaternions

Note that:

$$V^{(0)2} + V^{(1)}V^{(1)*} + V^{(2)}V^{(2)*} + V^{(3)2}$$

$$= V_0^2 + V_X^2 + V_Y^2 + V_Z^2 = 4$$

$$-(15)$$

The quaternion normalization condition is

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$$

$$-(16)$$

The quaternions and Euler angles are related by:

$$\left.\begin{aligned} q_0 + i q_3 &= \cos\frac{\beta}{2} \exp\left(\frac{i}{2}(\alpha+\gamma)\right) \\ q_1 - i q_2 &= \sin\frac{\beta}{2} \exp\left(-\frac{i}{2}(\alpha-\gamma)\right) \end{aligned}\right\} \quad -(17)$$

and the spinor rotation in $SU(2)$ is:

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} q_0 + i q_3 & q_1 - i q_2 \\ -q_1 - i q_2 & q_0 - i q_3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}. \quad -(18)$$

There is a clear relation between the Cartan tetrad and Hamilton's quaternions. There is also a relation to the Pauli matrices:

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad -(19)$$

because:

$$\sigma_0 - i\sigma_X = \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \quad -(20)$$

i.e.: $q_0 = 1, \; q_1 = 0, \; q_2 = 1, \; q_3 = 0$ $\quad -(21)$

and:

$$\sigma_0 + i\sigma_Y = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad -(22)$$

i.e. $q_0 = 1, \; q_1 = 1, \; q_2 = 0, \; q_3 = 0$ $\quad -(23)$

6)

Beltrami Topology

This is a Cartan topology if the tetrads of eq (2) are re-defined with a phase factor:

$$\underline{v}^{(1)} = \frac{1}{\sqrt{2}}\left(\underline{i} - i\underline{j}\right) e^{i(\omega t - \kappa z)} \quad - (24)$$

$$\underline{v}^{(2)} = \frac{1}{\sqrt{2}}\left(\underline{i} + i\underline{j}\right) e^{-i(\omega t - \kappa z)} \quad - (25)$$

$$\underline{v}^{(3)} = \underline{k} \quad - (26)$$

The same cyclic relations (1) apply.

In addition:

$$\underline{\nabla} \times \underline{v}^{(1)} = -\kappa \underline{v}^{(1)} \quad - (27)$$

$$\underline{\nabla} \times \underline{v}^{(2)} = \kappa \underline{v}^{(2)} \quad - (28)$$

$$\underline{\nabla} \times \underline{v}^{(3)} = \kappa_1 \underline{v}^{(3)} \quad - (29)$$

where:

$$\kappa_1 = 0. \quad - (30)$$

The Beltrami eigenvalues are $-\kappa$, κ and 0 in the eigen-equations:

7)

$$\underline{\nabla} \times \left(\underline{v}^{(1)} \right) = -\kappa \underline{v}^{(1)} \quad -(31)$$

$$\underline{\nabla} \times \left(\underline{v}^{(2)} \right) = \kappa \underline{v}^{(2)} \quad -(32)$$

$$\underline{\nabla} \times \left(\underline{v}^{(3)} \right) = 0 \underline{v}^{(3)} \quad -(33)$$

The eigenoperator is $\underline{\nabla} \times$ and the eigenfunctions are $\underline{v}^{(1)}$, $\underline{v}^{(2)}$ and $\underline{v}^{(3)}$. The eigenvalues $\kappa(-1, 0$ and $1)$ are those of a <u>boson</u>. The eigenvalues of a <u>fermion</u> are $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Adding the phase factor to the analysis (4) to (7) we obtain <u>one frame rotating and translating with respect to the other</u>, or <u>vice-versa</u>. So:

$$V^{a} = \begin{bmatrix} \frac{1}{\sqrt{2}}(1-i)e^{i\phi} \\ \frac{1}{\sqrt{2}}(1+i)e^{i\phi} \\ 1 \end{bmatrix} \quad -(34)$$

$$\text{if} \quad V^{\mu} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \quad -(35)$$

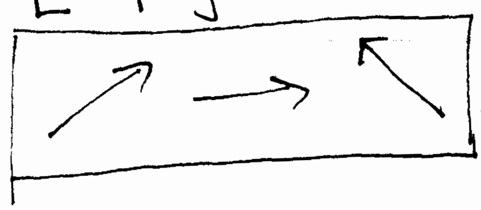
The real part of $\bar{V}^a$ is:

$$\text{Real } \bar{V}^a = \begin{bmatrix} \frac{1}{\sqrt{2}}(\cos\phi + \sin\phi) \\ \frac{1}{\sqrt{2}}(\cos\phi - \sin\phi) \\ 1 \end{bmatrix} \quad - (36)$$

Therefore:

$$\text{if } \phi = 0, \quad \bar{V}^a = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 1 \end{bmatrix}, \quad - (37)$$

$$\text{if } \phi = \frac{\pi}{2}, \quad \bar{V}^a = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 1 \end{bmatrix}, \quad - (38)$$

and so on, i.e. in 2-D: .

<u>Heaviside Topology</u>

In this case:

$$a = \mu \quad - (39)$$

and

$$q^a_\mu = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad - (40)$$

giving:

$$\underline{q} \times \underline{q} = \underline{0}. \quad - (41)$$