71(1): Lorentz and Coordinate Transformations

The most general type of coordinate transformation is given by Caroll, eq. 2.19, denoted (C2.19):

$$T^{\mu_1'\dots\mu_k'}{}_{\nu_1'\dots\nu_l'} = \left(\frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}}\dots\frac{\partial x^{\mu_k'}}{\partial x^{\mu_k}}\right)\left(\frac{\partial x^{\nu_1}}{\partial x^{\nu_1'}}\dots\frac{\partial x^{\nu_l}}{\partial x^{\nu_l'}}\right)T^{\mu_1\dots\mu_k}{}_{\nu_1\dots\nu_l} \quad -(1)$$

and the most basic property of general relativity is that a tensor transform as in eq. (1). A tensor in one frame must be a tensor in another frame, i.e. must transform covariantly. In Riemann geometry the Christoffel connection transforms as in (C3.6):

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\Gamma^{\mu}_{\nu\lambda} - \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\frac{\partial^2 x^{\nu'}}{\partial x^{\mu}\partial x^{\lambda}} \quad -(2)$$

and because of the second term on the RHS, does not transform as a tensor. The Christoffel connection is not therefore a tensor.

The spin connection on the other hand transforms as a vector as follows:

$$\omega^{a}{}_{\mu' b} = \left(\frac{\partial x^{\mu}}{\partial x^{\mu'}}\right)\omega^{a}{}_{\mu b} \quad -(3)$$

2) and the tetrad transforms as:

$$q^a_{\mu'} = \left(\frac{\partial x^\mu}{\partial x^{\mu'}} \right) q^a_\mu . \quad -(4)$$

For pure rotation:

$$\omega^a_{\mu b} = -\frac{\kappa}{2} \epsilon^a{}_{bc} \, q^c_\mu \quad -(5)$$

So:

$$\boxed{\omega^a_{\mu' b} = -\frac{\kappa}{2} \epsilon^a{}_{bc} \, q^c_{\mu'}} \quad -(6)$$

This means that the equation (5) is <u>generally covariant</u> as required by general relativity. This is a key equation for deriving the $\underline{B}^{(3)}$ field and is a true tensorial equation. <u>It can be constructed only in Cartan geometry</u>.

However, if we apply the Lorentz transform to the spin connection (C3.134):

$$\omega^{a'}_{\mu b'} = \Lambda^{a'}{}_a \Lambda^b{}_{b'} \omega^a_{\mu b} - \Lambda^c{}_{b'} \partial_\mu \Lambda^{a'}{}_c \quad -(7)$$

and $$q^{a'}_\mu = \Lambda^{a'}{}_a q^a_\mu \quad -(8)$$

So eq. (6) under the Lorentz transform is not the same in form (i.e. is not covariant) unless:

$$\Lambda^{c}{}_{b'}\,\partial_\mu \Lambda^{a'}{}_{c} = 0. \quad -(9)$$

This again shows that general relativity is needed to define the $\underline{B}^{(3)}$ field. The defining eq. (6) is generally covariant, but is not Lorentz covariant.

The other main advantage of using the spin connection is that covariant derivatives of spinors can be defined (Carroll, p. 91).

Every equation of Cartan geometry is generally covariant, so every equation of ECE theory is also generally covariant. The discussion of gauges and gauge constraints in general relativity given by Carroll on pages 122 ff can also be used in Cartan geometry. This will be the main subject of paper 71.