1) 70(1)a: Chirality and Spin in General Relativity

We first review the development in chapter 6 of volume 1, by defining the Cartesian vector in the standard notation:

$$\underline{R} = X\,\underline{i} + Y\,\underline{j} + Z\,\underline{k} . \quad -(1)$$

In the SU(2) basis:

$$R = \underline{\sigma} \cdot \underline{R} = X\sigma_1 + Y\sigma_2 + Z\sigma_3$$

$$= \begin{bmatrix} Z & X - iY \\ X + iY & -Z \end{bmatrix} \quad -(2)$$

so that:

$$R^2 = (X^2 + Y^2 + Z^2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . \quad -(3)$$

The SU(2) group is that of the unitary, unimodular matrices:

$$UU^+ = \underline{1}, \quad \det U = 1 . \quad -(4)$$

These have the general form:

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \quad -(5)$$

with

$$aa^* + bb^* = 1 . \quad -(6)$$

Define the two-component spinor with complex valued elements:

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \quad -(7)$$

$$\xi^+ = [\xi_1^*, \xi_2^*] . \quad -(8)$$

We obtain the invariant:

2)

$$X^2 + Y^2 + Z^2 = \xi_1 \xi_1^* + \xi_2 \xi_2^* \quad -(9)$$

Chirality Column Vector in SU(2)

This is defined by:

$$\xi_1 = \frac{1}{\sqrt{2}}(1 - i) \quad -(10)$$

$$\xi_2 = \frac{1}{\sqrt{2}}(1 + i) \quad -(11)$$

and:

$$V^a = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} := \begin{bmatrix} e^R \\ e^L \end{bmatrix} \quad -(12)$$

as in eq. (13.24) of volume 2.

Spin Column Vector in SU(2)

This is defined by:

$$V^\mu = e^{-i\phi} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad -(13)$$

where ϕ is the phase of the fermion field.

The chirality and spin column vectors are related by the SU(2) tetrad field:

$$V^a = q^a_\mu V^\mu, \quad -(14)$$

$$q^a_\mu = \frac{e^{i\phi}}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \quad -(15)$$

3) The tetrad (15) is a simple example of a right or left handed spinning field. It can be seen that:

$$\xi_1 \xi_2 = 1 = \frac{1}{\sqrt{2}}(1-i)\frac{1}{\sqrt{2}}(1+i) \quad -(16)$$

so the origin of chirality, or left and right handedness, is the factorization in equation (16).

If:

$$\phi = \omega t - \kappa Y \quad -(17)$$

it is seen that:

$$\Box q^a_\mu = 0 \quad -(18)$$

and this is the equation of the hypothetical massless fermion. If the two rows of the tetrad matrix in eq. (15) are transposed into columns:

$$\phi := q^{a\,T}_\mu := \frac{e^{i\phi}}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \\ 1 \\ i \end{bmatrix}. \quad -(19)$$

The four-spinor ϕ consists of two Pauli spinors:

$$\phi^R = \frac{e^{i\phi}}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}, \quad \phi^L = \frac{e^{i\phi}}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix} \quad -(20)$$

and

$$\Box \phi = 0. \quad -(21)$$

4) Denote the Y Pauli matrix by:

$$\sigma_Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad -(22)$$

to obtain the Weyl equations:

$$\sigma_Y \phi^R = -\phi^R \quad -(23)$$

$$\sigma_Y \phi^L = \phi^L \quad -(24)$$

i.e.

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} -1 \\ i \end{bmatrix} = -\begin{bmatrix} 1 \\ -i \end{bmatrix} \quad -(23a)$$

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix}. \quad -(23b)$$

The helicity eigenvalues in Eqs. (23) and (24) are ± 1. The eigenfunctions are the right and left Pauli spinors ϕ^R and ϕ^L and the eigenoperator is σ_Y. Therefore the Pauli spinors ϕ^R and ϕ^L are those of a massless fermion propagating along Y.

This is a simple example of how chirality or helicity can be built up from two types of column vector.

5) This is an example in special relativity, because the Weyl equations are the massless Dirac equations. In general relativity:

$$\begin{bmatrix} e^R \\ e_L \end{bmatrix} = \begin{bmatrix} q^R_1 & q^R_2 \\ q^L_1 & q^L_2 \end{bmatrix} \begin{bmatrix} q^1 \\ q^2 \end{bmatrix} \quad -(24)$$

and

$$(\square + kT)\, q^a_\mu = 0 \quad -(25)$$

where:

$$q^a_\mu = \begin{bmatrix} q^R_1 & q^R_2 \\ q^L_1 & q^L_2 \end{bmatrix}. \quad -(26)$$

If we define:

$$\psi := \begin{bmatrix} q^R_1 \\ q^R_2 \\ q^L_1 \\ q^L_2 \end{bmatrix} \quad -(27)$$

then:

$$\boxed{(\square + kT)\, \psi = 0} \quad -(28)$$

It is known experimentally that this must reduce to the Dirac equation for the free single fermion:

$$\left(\Box + \left(\frac{mc}{\hbar} \right)^2 \right) \phi = 0 \quad -(29)$$

Eq. (29) is true in the absence of gravitation, in which case:

$$kT = \frac{m^2 c^2}{\hbar^2} \quad -(30)$$

Eq. (30) is an example of the equivalence principle. The effect of gravitation on a fermion is given by eq. (28). Therefore eq. (28) describes two interacting electrons. It is therefore equivalent to the Coulomb law, and ~~thus~~ shows that electrodynamics can be described by SU(2) representation space algebra. This was first shown by Majorana in the twenties.

This is another way of showing that ECE theory is a true unified field theory that makes many profound and new predictions