1)

Notes 64(4): Resonant Coulomb Law for Any Driving Force

This is:

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - \frac{1}{r^2}\phi = -\frac{\rho(0)}{\epsilon_0} f(\kappa r) \quad -(1)$$

where $f(\kappa r)$ is any suitable function. Now use:

$$\kappa r = \exp(i\kappa R) \quad -(2)$$

to find:

$$\frac{d^2\phi}{dR^2} + \kappa^2\phi = \frac{\rho(0)}{\epsilon_0} e^{2i\kappa R} f\left(e^{i\kappa R}\right). \quad -(2)$$

Assume ~~that~~:

$$\phi = A\frac{\rho(0)}{\epsilon_0} e^{2i\kappa R} f\left(e^{i\kappa R}\right) \quad -(3)$$

~~then~~:

$$\frac{d\phi}{dR} = i\kappa A\frac{\rho(0)}{\epsilon_0}\left(2e^{2i\kappa R} f\left(e^{i\kappa R}\right) + e^{3i\kappa R} f'\left(e^{i\kappa R}\right)\right) \quad -(4)$$

and:

$$\frac{d^2\phi}{dR^2} = -\kappa^2 A\frac{\rho(0)}{\epsilon_0}\left(4e^{2i\kappa R} f + 5e^{3i\kappa R} f' + e^{4i\kappa R} f''\right) \quad -(5)$$

From eqs (2) to (5):

2)

$$A = -\frac{1}{k^2}\left(\frac{e^{2ikR}f}{3e^{2ikR}f + 5e^{3ikR}f' + e^{4ikR}f''}\right) \quad -(6)$$

so:

$$\phi = -\frac{\rho(0)}{\epsilon_0 k^2}\left(\frac{e^{4ikR}f^2}{3e^{2ikR}f + 5e^{3ikR}f' + e^{4ikR}f''}\right)$$

$$\boxed{\phi = -\frac{\rho(0)}{\epsilon_0}\left(\frac{r^2 f^2}{3f + 5krf' + k^2r^2 f''}\right)} \quad -(7)$$

Examples of Notation

1) If $f = \cos(x)$,

$$f' = -\sin(x)$$

$$f'' = -\cos(x) \quad -(8)$$

$$\phi = \frac{\rho(0)}{\epsilon_0}\left(\frac{r^2\cos^2(kr)}{5kr\sin(kr) + k^2r^2\cos(kr) - 3\cos(kr)}\right)$$

giving three resonance peaks.

2) If $f = 1$, $f' = 0$, $f'' = 0$

)
$$\phi = -\frac{\rho(0)}{\epsilon_0}\frac{r^2}{3} \quad -(9)$$

$$= -\frac{e}{4\pi\epsilon_0 r} \quad -(10)$$

if $\rho(0) = \frac{e}{V}$, $V = \frac{4}{3}\pi r^3$ $\quad -(11)$

This is the Coulomb law far off resonance.

3) If
$$f = e^{ix} \quad -(12)$$
$$f' = ie^{ix} \quad -(13)$$
$$f'' = -e^{ix} \quad -(14)$$

$$\phi = -\frac{\rho(0)}{\epsilon_0}\left(\frac{r^2 e^{2ik^2r^2}}{3e^{ikr} + i5kre^{ikr} - k^2r^2e^{ikr}}\right) \quad -(15)$$

4) If
$$f = e^{-x} \quad -(16)$$
$$f' = -e^{-x} \quad -(17)$$
$$f'' = e^{-x} \quad -(18)$$

$$\phi = -\frac{\rho(0)}{\epsilon_0}\left(\frac{r^2 e^{-2kr}}{3e^{-kr} - 5kre^{-kr} + k^2r^2e^{-kr}}\right) \quad -(19)$$

4) In this case:

$$\phi = -\frac{\rho(0)}{\epsilon_0} r^2 \left(\frac{e^{-\kappa r}}{3 - 5\kappa r + \kappa^2 r^2} \right) \quad -(20)$$

<u>Resonance</u> occurs when:

$$x^2 - 5x + 3 = 0 \quad -(21)$$

$$x = \frac{1}{2}\left(5 \pm (25-12)^{1/2}\right)$$

$$= \frac{1}{2}\left(5 \pm \sqrt{13}\right)$$

$$\boxed{x = 4.3028 \text{ and } 0.6972} \quad -(22)$$

Using: $1 \text{ radian} = 57.296^\circ \quad -(23)$

then $\boxed{x = 246.53^\circ \text{ and } 39.947^\circ}$

giving <u>two resonance peaks for an exponentially decaying driving force.</u>

5) If $f = e^{-x}\cos x \quad -(24)$

$$f' = e^{-x}\cos x - e^{-x}\sin x$$

$$= e^{-x}(\cos x - \sin x)$$

5)

$$f'' = e^{-x}\left(-\sin x + \cos x - \sin x - \cos x\right)$$
$$= -2e^{-x}\sin x$$

so:

$$\phi = -\frac{\rho(0)}{\epsilon_0}\left(\frac{r^2\cos^2(kr)\,e^{-2kr}}{3e^{-kr}\cos(kr) + 5kr\left(e^{-kr}\cos(kr) - e^{-kr}\sin(kr)\right) + 6e^{-kr}\sin(kr)k^2r^2}\right)$$

$$= -\frac{\rho(0)}{\epsilon_0}\,\frac{r^2 e^{-kr}\cos^2(kr)}{\left(3\cos(kr) + 5kr\left(\cos(kr) - \sin(kr)\right) + 6k^2r^2\sin(kr)\right)}$$

Resonance occurs at:

$$(3 + 5x)\cos x + 6x^2\sin x - 5x\sin x = 0$$

$$\boxed{\tan x = \frac{3 + 5x}{x(5 - 6x)}}$$