Paper 63: Checking Calculations (Notes 63(25))

The initial equation is:

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - \frac{1}{r^2}\phi = -\frac{\rho(0)}{\epsilon_0}\cos(\kappa r). \quad -(1)$$

with the change of variable:

$$\kappa r = \exp(i\kappa R) \quad -(2)$$

eq (1) becomes:

$$\frac{d^2\phi}{dR^2} + \kappa^2\phi = \frac{\rho(0)}{\epsilon_0} e^{2i\kappa R}\cos\left(e^{i\kappa R}\right). \quad -(2a)$$

Assume a solution of the type:

$$\phi = \frac{A\rho(0)}{\epsilon_0} e^{2i\kappa R}\cos\left(e^{i\kappa R}\right) \quad -(3)$$

$$\frac{d\phi}{dR} = 2i\kappa\frac{A\rho(0)}{\epsilon_0} e^{2i\kappa R}\cos\left(e^{i\kappa R}\right) - i\kappa e^{i\kappa R}\sin\left(e^{i\kappa R}\right)\frac{A\rho(0)}{\epsilon_0}e^{2i\kappa R}$$

$$= i\kappa\frac{A\rho(0)}{\epsilon_0}\left(2e^{2i\kappa R}\cos\left(e^{i\kappa R}\right) - e^{3i\kappa R}\sin\left(e^{i\kappa R}\right)\right)$$

$$\frac{d^2\phi}{dR^2} = i\kappa\frac{A\rho(0)}{\epsilon_0}\left(4ie^{2i\kappa R}\cos\left(e^{i\kappa R}\right) - 2i\kappa e^{3i\kappa R}\sin\left(e^{i\kappa R}\right) - 3i\kappa e^{3i\kappa R}\sin\left(e^{i\kappa R}\right) - e^{4i\kappa R}\cos\left(e^{i\kappa R}\right)\right)$$

$$= \frac{A\rho(0)}{\epsilon_0}\left(5\kappa^2 e^{3i\kappa R}\sin\left(e^{i\kappa R}\right) + \kappa^2 e^{4i\kappa R}\cos\left(e^{i\kappa R}\right) - 4\kappa^2 e^{2i\kappa R}\cos\left(e^{i\kappa R}\right)\right) \quad -(4)$$

2) Using eqs (3) and (4) in eq (2a):

$$A\,k^2\left(5\,e^{3ikR}\sin\left(e^{ikR}\right)+e^{4ikR}\cos\left(e^{ikR}\right)-3\,e^{2ikR}\cos\left(e^{ikR}\right)\right) \qquad -(5)$$

$$= e^{2ikR}\cos\left(e^{ikR}\right)$$

$$A = \frac{1}{k^2}\frac{e^{2ikR}\cos\left(e^{ikR}\right)}{\left(5e^{3ikR}\sin\left(e^{ikR}\right)+e^{4ikR}\cos\left(e^{ikR}\right)-3e^{2ikR}\cos\left(e^{ikR}\right)\right)}$$

$$= \frac{k^2r^2\cos(kr)}{k^2\left(5k^3r^3\sin(kr)+k^4r^4\cos(kr)-3k^2r^2\cos(kr)\right)}$$

$$\boxed{A = \frac{\cos(kr)}{k^2\left(5kr\sin(kr)+k^2r^2\cos(kr)-3\cos(kr)\right)}}. \qquad -(6)$$

Using eq (6) in eq. (3):

$$\boxed{\phi = \frac{\rho(0)}{\epsilon_0}\frac{r^2\cos^2(kr)}{\left(5kr\sin(kr)+k^2r^2\cos(kr)-3\cos(kr)\right)}} \qquad -(7).$$

Eq (7) is the solution of eq. (1).

3)

Limits of eq (7)

1) **Resonance** occurs when:

$$x^2 \cos x + 5x \sin x - 3\cos x = 0 \quad -(8)$$

where:

$$x := \kappa r. \quad -(9)$$

Eq (8) is:

$$\boxed{\tan x = \frac{3 - x^2}{5x}}. \quad -(10)$$

The solution to within 0.5% is:

$$x = 0.661 = 37.9^\circ. \quad -(11)$$

Therefore:

$$\boxed{\phi \rightarrow \infty \text{ when } \kappa r = 0.661}. \quad -(12)$$

2) The $\kappa \rightarrow 0$ limit means that eq. (2) becomes:

$$\frac{d^2\phi}{dR^2} = \frac{\rho(0)}{\epsilon_0}. \quad -(13)$$

Eq (7) gives this limit as follows:

$$\phi = \frac{\rho(0)}{\epsilon_0} \left(\frac{\cos^2(\kappa r)}{5\frac{\kappa}{r}\sin(\kappa r) + \kappa^2\cos(\kappa r) - \frac{3}{r^2}\cos(\kappa r)} \right)$$

$$\xrightarrow[\kappa \rightarrow 0]{} -\frac{\rho(0)}{\epsilon_0}\frac{r^2}{3} \quad -(14)$$

4) Eq (14) is self-consistently the solution to:

$$\frac{d^2\phi}{dr^2} = \frac{-\rho^{(0)}}{3\epsilon_0} \quad -(15)$$

$$:= -\frac{\rho}{\epsilon_0}.$$

Equation (15) is the Poisson equation. Therefore in the $\kappa \to 0$ limit eq. (1) reduces to eq (15), and gives the Coulomb potential off resonance:

$$\phi = \frac{e}{4\pi\epsilon_0 r}. \quad -(16)$$

Eq (16) gives the off-resonance spin connection:

$$\underline{\nabla}\phi = \underline{\omega}\phi \quad -(17)$$

$$\underline{\omega} = -\frac{1}{r}\underline{e}_r \quad -(18)$$

where $\underline{e}_r$ is the radial unit vector. The modulus of the spin connection is:

$$|\underline{\omega}| = 1/r. \quad -(19)$$

In deriving eq (1) it has been assumed that this spin connection is also true under all conditions, i.e. is a basic property of spacetime.