1)

61: Notes on Interaction of Fields

The general rule in field theory to describe the interaction of fields is to replace the derivative by the covariant derivative. In Cartan geometry this rule becomes:

$$d\wedge \rightarrow D\wedge = d\wedge + \omega\wedge \quad -(1)$$

where $d\wedge$ is the exterior derivative and $D\wedge$ is the covariant exterior derivative. The latter is defined in terms of the spin connection ω. The ECE wave equation already incorporates this rule because the former is derived from:

$$D^\mu \left(D_\mu q^a_\sim \right) := 0. \quad -(2)$$

This is an identity of Cartan geometry derived from the tetrad postulate:

$$D_\mu q^a_\sim = 0. \quad -(3)$$

Eq (2) is equivalent to:

$$\square q^a_\sim = R q^a_\sim \quad -(4)$$

where:

$$R := q^\lambda_a \partial^\mu \left(\Gamma^\sim_{\mu\lambda} q^a_\sim - \omega^a_{\mu b} q^b_\lambda \right). \quad -(5)$$

The fundamental assumption of general relativity is

$$R = -kT \quad -(6)$$

and so:

$$\boxed{\left(\square + kT \right) q^a_\sim = 0} \quad -(7)$$

2)

Eq. (7) is valid for all fields, gravitational, electromagnetic, weak, strong, fermions and bosons. Therefore in eq. (7) the interaction of fields is described by kT, because the covariant derivative D_μ has already replaced the derivative ∂_μ. In other words the minimal prescription is already included in eq. (7), which is therefore capable of describing the interaction of any field with any other field or combination of fields.

Eq. (6) was originally derived by Einstein from the Einstein / Hilbert field equation for gravitation. However, if R is defined as in eq. (5), eq. (6) is valid for all fields. Here:

$$k = \frac{8\pi G}{c^2} = 1.86595 \times 10^{-26}\ N s^2 kg^{-2}$$

$$G = 6.6726 \times 10^{-11}\ N m^2 kg^{-2}$$

Here:

$$N = \text{newtons} = kg\ m\ s^{-2}$$

so:

$$\boxed{k = m\ kg^{-1} = \text{metres / kilogram}}$$

The units of R are m^{-2}, so:

$$\boxed{T = kg\ m^{-3} = \text{kilograms / metre}^3}$$

The units of T are therefore:

$$T = \frac{1}{c^2}\left(\frac{\text{joules}}{\text{volume}}\right) = \frac{1}{c^2}\left(\frac{\text{energy}}{\text{volume}}\right)$$

3)

If we consider a free electron, it does not interact with any other field, and in the limit of special relativity it is described by the Dirac equation:

$$\left(\square + \left(\frac{mc}{\hbar}\right)^2 \right) q_\mu^a = 0. \quad -(2)$$

In this limit, the Einstein equivalence principle states that it must be free of gravitation, and more generally, any other field. So in this limit:

$$kT = \frac{m^2c^2}{\hbar^2}. \quad -(3)$$

The rest energy of the electron is:

$$E_0 = mc^2, \quad -(4)$$

so the rest value of T is defined as:

$$T_0 := \frac{m}{V_0} \quad -(5)$$

where V_0 is the rest volume. So from eqns. (3) and (5):

$$k\frac{m}{V_0} = \frac{m^2c^2}{\hbar^2} \quad -(6)$$

i.e.

$$\boxed{V_0 = \frac{k\hbar^2}{mc^2} = \frac{k\hbar^2}{E_0}} \quad -(7)$$

This result was first derived in volume 1 of M.W. Evans, "Generally Covariant Unified Field Theory"

4) In all frames in classical special relativity:

$$p^\mu p_\mu = m^2 c^2. \quad -(8)$$

This is the Einstein equation of special relativity. In the rest frame, eq. (8) becomes eq. (4). Using the operator equivalence:

$$p^\mu = i\hbar \partial^\mu \quad -(9)$$

we find: $$m^2 c^2 = p^\mu p_\mu = -\hbar^2 \Box \quad -(10)$$

i.e.: $$\frac{m^2 c^2}{\hbar^2} = -\Box \quad -(11)$$

and eq. (2) becomes:

$$(\Box - \Box) q^a_\mu := 0. \quad -(12)$$

Eq. (8) is:

$$E_n^2 = p^2 c^2 + m^2 c^4 \quad -(13)$$
$$= p^2 c^2 + E_0^2 \quad -(14)$$

Therefore in general:

$$T = \pm \frac{1}{V} \cdot \frac{1}{c^2} \left(p^2 c^2 + E_0^2 \right)^{1/2}$$
$$-(15)$$

so $$\frac{k}{Vc^2} \left(p^2 c^2 + E_0^2 \right)^{1/2} = \frac{m^2 c^2}{\hbar^2} \quad -(16)$$

5) and :

$$\boxed{\bar{V} = \frac{\pounds^2 k}{m^2 c^4} \left(p^2 c^2 + E_0^2 \right)^{1/2}} \quad -(17)$$

In eq. (17), p is defined as the special relativistic momentum:

$$p = \gamma m v \quad -(18)$$

where:

$$\gamma = \left(1 - \frac{v^2}{c^2} \right)^{-1/2} \quad -(19)$$

So if

$$v \rightarrow c \quad -(20)$$

the volume $\bar{V}$ becomes infinite, and the special relativistic energy:

$$E_n = \gamma m c^2 \quad -(21)$$

also becomes infinite. The ratio of E_n to $\bar{V}$ is however constant:

$$\boxed{\frac{E_n}{\bar{V}} = \frac{m^2 c^4}{k \pounds^2}} \quad -(22)$$

i.e.

$$\bar{V} = \frac{1}{k \pounds^2} \frac{E_n}{m^2 c^4} \quad -(23)$$

which reduces to eq. (7) if :

$$E_n = E_0 . \quad -(24)$$

From eqs. (21) and (22) :

$$\gamma = \bar{V} \frac{m c^2}{k \pounds^2} \quad -(25)$$

6) and so:

$$V = \gamma \left(\frac{k\hbar^2}{mc^2} \right) \quad -(26)$$

$$\boxed{V = \gamma V_0} \quad -(27)$$

Eq. (27) means that:

$$V_0 = \left(1 - \frac{v^2}{c^2}\right)^{1/2} V \quad -(28)$$

$$= \frac{k\hbar^2}{mc^2}.$$

This is the (Fitzgerald) contraction of volume to the rest volume V_0 of the particle. As $v \to c$, $V \to \infty$ to keep V_0 constant for given m.

These considerations apply to a free particle. If the particle interacts with any field or combination of fields, then $(mc/\hbar)^2$ is replaced by kT. This means that the Compton wavelength:

$$\lambda_c = \frac{\hbar}{mc} \quad -(29)$$

is changed by field interaction. A free particle exists only in special relativity, where there is no acceleration of any kind.

7)

In the usual minimal prescription method to describe the interaction of the electron with an electromagnetic potential field A_μ:

$$\partial_\mu \rightarrow \partial_\mu - i \frac{e}{\hbar} A_\mu . \quad —(30)$$

Therefore:

$$\Box \rightarrow \left(\partial^\mu - i\frac{e}{\hbar}A^\mu\right)\left(\partial_\mu - i\frac{e}{\hbar}A_\mu\right), \quad -(31)$$

or for complex valued A^μ:

$$\Box \rightarrow \Box + \frac{e^2}{\hbar^2} A^\mu A_\mu^* + \ldots \quad —(32)$$

and in eq. (2):

$$\left(\Box + \frac{e^2}{\hbar^2} A^\mu A_\mu^* + \ldots + \frac{m^2c^2}{\hbar^2}\right) q_\nu^a = 0 \quad —(33)$$

So from comparison of eqns. (7) and (33):

$$kT = \frac{m^2c^2}{\hbar^2} + \frac{e^2}{\hbar^2} A^\mu A_\mu^* + \ldots \quad —(34)$$

and this illustrates that kT describes the interaction of the electron and electromagnetic potential field.