Eddington on the Deflection of Light

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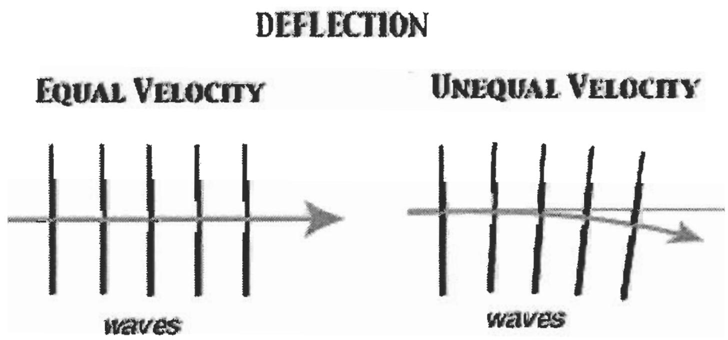
Arthur Stanley Eddington was a good writer as well as a good scientist, and his popular book, *Space, Time and Gravitation* (1920) is still worth reading even today. Since he explains the deflection of starlight (he confirmed this in the1919 British expeditions) so nicely, I cannot refrain from quoting the relevant part (for the following calculation, Schwarzschild coordinates are used, and mass m is measured in units of length. See Schwarzschild Geometry, where M is used instead of m).

> The wave-motion in a ray of light can be compared to a succession of long straight waves rolling onward in the sea. If the motion of the waves is slower at one end than the other, the whole wave-front must gradually slew round, and the direction in which it is rolling must change. In the sea this happens when one end of the wave reaches shallow water before the other, because the speed in shallow water is slower. It is well known that this causes waves proceeding diagonally across a bay to slew ound and come in parrallel to the shore; the advanced end is delayed in the shallow water and waits for the other. In the same way when the light waves pass near the sun, the end nearest the sun has the smaller velocity and the wave-front slews round; thus the course of the waves is bent.



> Light moves more slowly in a material medium than in vacuum, the velocity being inversely proportional to the refractive index of the medium. The phenomenon of refraction is in fact caused by a slewing of the wave-front in passing into a region of smaller velocity. We can thus imitate the gravitational effect on light precisely, if we imagine the space round the sun filled with a refracting medium which gives the appropriate velocity of light. To give the velocity $1 - 2m/r$, the refractive index must be $1/(1 - 2m/r)$, or, very

approximately, $1 + 2m/r$. At the surface of the sun, r = 697,000km., m = 1.47 km., hence the necessary refractive index is 1.00000424. At a height above the sun equal to the radius it is 1.00000212.

Any problem on the paths of rays near the sun can now be solved by the methods of geometrical optics applied to the equivalent refracting medium. It is not difficult to show that the total deflection of a ray of light passing at a distance r from the center the sun is (in circular measure)

$$4m/r,$$

whereas the deflection of the same ray calculated on the Newtonian theory would be

$$2m/r.$$

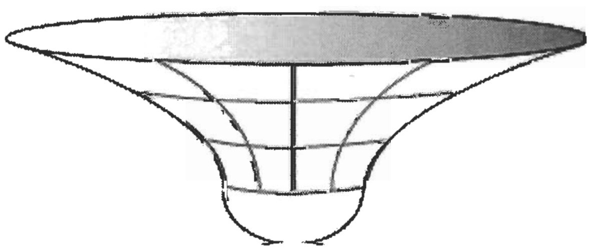
For a ray grazing the surface of the sun the numerical value of this deflection is

1".75 (Einstein's theory),

0".87 (Newtonian theory).

(Eddington 1920, 108-9.)

Recall that Einstein's theory demands *spacetime* curvature, which means that *both* space and time are warped. Earlier (in 1911; see Genesis of General Relativity (2)), Einstein considered only the time warps and obtained only a half of the preceding deflection; beginning 1912, he realized that space is also warped, and he envisaged a non-Euclidean geometry for treating gravity. On the other hand, the usual Embedding Diagram shows only the spatial curvature (time is frozen), so that you have to add time warps, in order to obtain the correct curvature. The following figure is an embedding diagram of an equatorial plane of the sun (white part is the interior, and the rest is the exterior space, of the sun).



Try to learn physics or any other disciplines, whenever a related topic appear in the course of reading (that's the best way to do philosophy of *science*)!

Reference

Eddington, A. S. (1920) *Space, Time and Gravitation*, Cambridge University Press, 1987.

Phil. spacetime // BACK TO ERROR STATISTICS

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6.3 Bending Light

At the conclusion of his treatise on Opticks in 1704, the (then) 62 year old Newton lamented that he could "not now think of taking these things into farther consideration", and contented himself with proposing a number of queries "in order to a farther search to be made by others". The very first of these was

> Do not Bodies act upon Light at a distance, and by their action bend its Rays, and is not this action strongest at the least distance?

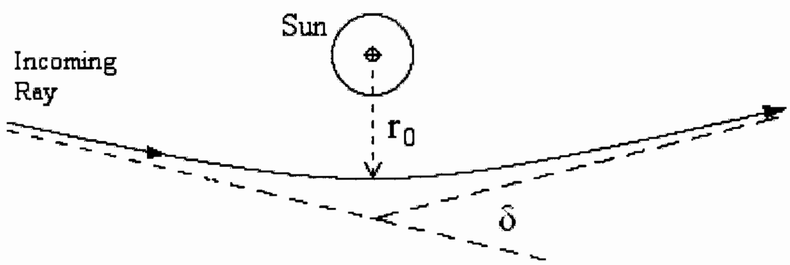
Superficially this may not seem like a very radical suggestion, because on the basis of the corpuscular theory of light, and Newton's laws of mechanics and gravitation, it's easy to conjecture that a beam of light might be deflected slightly as it passes near a large massive body, assuming particles of light respond to gravitational acceleration similarly to particles of matter. For any conical orbit of a small test particle in a Newtonian gravitational field around a central mass m, the eccentricity is given by

$$\varepsilon = \sqrt{1+\frac{2Eh^2}{m^2}}$$

where $E = v^2/2 - m/r$ is the total energy (kinetic plus potential), $h = rv_t$ is the angular momentum, v is the total speed, v_t is the tangential component of the speed, and r is the radial distance from the center of the mass. Since a beam of light travels at such a high speed, it will be in a shallow hyperbolic orbit around an ordinary massive object like the Sun. Letting r_0 denote the closest approach (the perihelion) of the beam to the gravitating body, at which $v = v_t$, we have

$$\varepsilon = \sqrt{1+\left(\frac{r_0 v^2}{m}\right)^2 - \frac{2r_0 v^2}{m}} = \left(\frac{r_0 v^2}{m} - 1\right)$$

Now we set v = 1 (the speed of light in geometric units) at the perihelion, and from the geometry of the hyperbola we know that the asymptotes make an angle of α with the axis of symmetry, where $\cos(\alpha) = -1/\varepsilon$.



With a hyperbola as shown in the figure above, this implies that the total angular deflection of the beam of light is $\delta = 2(\alpha - \pi/2)$, which for small angles α and for m (in geometric units) much less than r_0 is given in Newtonian mechanics by

$$\delta = 2\left(\mathrm{acos}\left(\frac{m}{m - r_0}\right) - \frac{\pi}{2}\right) \approx \frac{2m}{r_0}$$

The best natural opportunity to observe this deflection would be to look at the stars near the perimeter of the Sun during a solar eclipse. The mass of the Sun in gravitational units is about m = 1475 meters, and a beam of light just skimming past the Sun would have a closest distance equal to the Sun's radius, $r = (6.95)10^8$ meters. Therefore, the Newtonian prediction would be 0.000004245 radians, which equals 0.875 seconds of arc. (There are 2π radians per 360 degrees, each of degree representing 60 minutes of arc, and each minute represents 60 seconds of arc.)

However, there is a problematical aspect to this "Newtonian" prediction, because it's based on the assumption that particles of light can be accelerated and decelerated just like ordinary matter, and yet if this were the case, it would be difficult to explain why (in non-relativistic absolute space and time) all the light that we observe is traveling at a single characteristic speed. Admittedly if we posit that the rest mass of a particle of light is extremely small, it might be impossible to interact with such a particle without imparting to it a very high velocity, but this doesn't explain why all light seems to have precisely the same velocity, as if this particular speed is somehow a characteristic property of light. As a result of these concerns, especially as the wave conception of light began to supersede the corpuscular theory, the idea that gravity might bend light rays was largely discounted in Newtonian physics. (The same fate befell the idea of black holes, originally proposed by Mitchell based on the Newtonian escape velocity for light. Laplace also mentioned the idea in his Celestial Mechanics, but deleted it in the third edition, possibly because of the conceptual difficulties discussed here.)

The idea of bending light was revived in Einstein's 1911 paper "On the Influence of Gravitation on the Propagation of Light". Oddly enough, the quantitative prediction given in this paper for the amount of deflection of light passing near a large mass was identical to the old Newtonian prediction, $\delta = 2m/r_0$. There were several attempts to measure the deflection of starlight passing close by the Sun during solar eclipses to test Einstein's prediction in the years between 1911 and 1915, but all these attempts were thwarted by cloudy skies, logistical problems, the First World War, etc. Einstein became very exasperated over the repeated failures of the experimentalists to gather any useful data, because he was eager to see his prediction corroborated, which he was certain it would be. Ironically, if any of those early experimental efforts had succeeded in collecting useful data, they would have proven Einstein *wrong*! It wasn't until late in 1915, as he completed the general theory, that Einstein realized his earlier prediction was incorrect, and the angular deflection should actually be *twice* the size he predicted in 1911. Had the World War not intervened, it's likely that Einstein would never have been able to claim the bending of light (at twice the Newtonian value) as a prediction of general relativity. At best he would have been forced to explain, after the fact, why the observed deflection was actually consistent with the completed general theory. (This would have made it somewhat similar the cosmological expansion, which would have been one of the most magnificent theoretical predictions in the history of science, but the experimentalist Hubble got there first.) Luckily for Einstein, he corrected the light-bending prediction before any expeditions succeeded in making useful observations. In 1919, after the war had ended, scientific expeditions were sent to Sobral in South America and Principe in West Africa to make observations of the solar eclipse. The reported results were angular deflections of 1.98 ± 0.16 and 1.61 ± 0.40 seconds of arc, respectively, which was taken as clear confirmation of general relativity's prediction of 1.75 seconds of arc. This success, combined with the esoteric appeal of bending light, and the romantic adventure of the eclipse expeditions themselves, contributed enormously to making Einstein a world celebrity.

One other intriguing aspect of the story, in retrospect, is the fact that there is serious doubt about whether the measurement techniques used by the 1919 expeditions were robust enough to have legitimately detected the deflections which were reported. Experimentalists must always be wary of the "Ouija board" effect, with their hands on the instruments, knowing what results they want or expect. This makes it especially interesting to speculate on what values would have been recorded if they had managed to take readings in 1914, when the expected deflection was still just 0.875 seconds of arc. (It should be mentioned that many subsequent observations, summarized below, have independently confirmed the angular deflection predicted by general relativity, i.e., twice the "Newtonian" value.)

To determine the relativistic prediction for the bending of light past the Sun, the conventional approach is to simply evaluate the solution of the four geodesic equations presented in Chapter 5.2, but this involves a three-dimensional manifold, with a large number of Christoffel symbols, etc. It's possible to treat the problem more efficiently by considering it from a two-dimensional standpoint. Recall the Schwarzschild metric in the usual polar coordinates

$$(d\tau)^2 = \left(\frac{r-2m}{r}\right)(dt)^2 - \left(\frac{r}{r-2m}\right)(dr)^2 - r^2(d\theta)^2 - r^2\sin(\theta)^2(d\phi)^2$$

We'll restrict our attention to a single plane through the center of mass by setting $\phi = 0$, and since light travels along null paths, we set $d\tau = 0$, which allows us to write the remaining terms in the form

$$(dt)^2 = \left(\frac{r}{r-2m}\right)^2(dr)^2 + \frac{r^3}{r-2m}(d\theta)^2 \qquad (1)$$

This can be regarded as the (positive-definite) line element of a two-dimensional surface (r, θ), with the parameter t serving as the metrical distance. The null paths satisfying the complete spacetime metric with $d\tau = 0$ are stationary if and only if they are stationary with respect to (1). This implies Fermat's Principle of "least time", i.e., light follows paths that minimize the integrated time of flight, or, more generally, paths for which the elapsed Schwarzschild coordinate time is stationary, as discussed in Chapter 3.5. (Equivalently, we have an angular analog of Fermat's Principle, i.e., light follows paths that make the angular displacement $d\theta$ stationary, because the coefficients of (1) are independent of both t and θ.) Therefore, we need only determine the geodesic paths on this surface. The covariant and contravariant metric tensors are simply

$$g_{\mu\nu} = \begin{bmatrix} \left(\frac{r}{r-2m}\right)^2 & 0 \\ 0 & \frac{r^3}{r-2m} \end{bmatrix} \qquad g^{\mu\nu} = \begin{bmatrix} \left(\frac{r-2m}{r}\right)^2 & 0 \\ 0 & \frac{r-2m}{r^3} \end{bmatrix}$$

and the only non-zero partial derivatives of the components of $g_{\mu\nu}$ are

$$\frac{\partial g_{rr}}{\partial r} = \frac{-4mr}{(r-2m)^3} \qquad \frac{\partial g_{\theta\theta}}{\partial r} = 2r^2 \frac{r-3m}{(r-2m)^2}$$

so the non-zero Christoffel symbols are

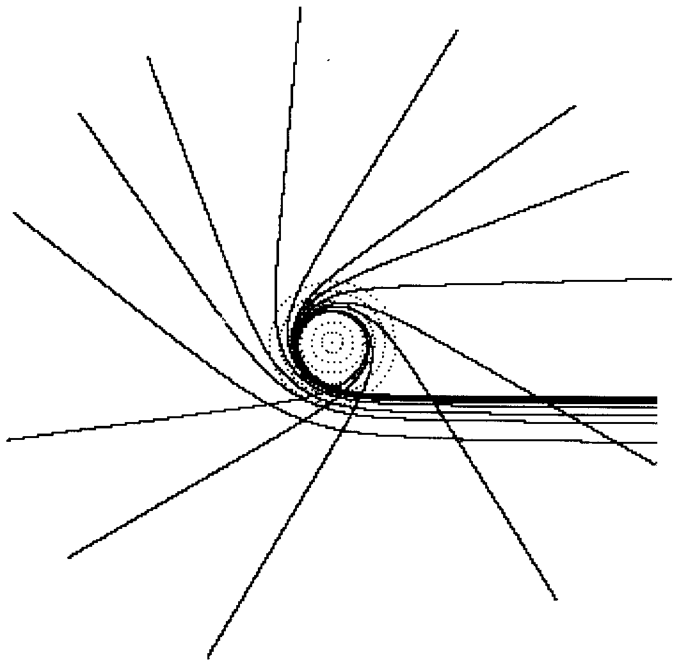
$$\Gamma^r{}_{rr} = \frac{-2m}{r(r-2m)} \qquad \Gamma^r{}_{\theta\theta} = -(r-3m) \qquad \Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \frac{r-3m}{r(r-2m)}$$

Taking the coordinate time t as the path parameter (since it plays the role of the metrical distance in this geometry), the two equations for geodesic paths on the (r, θ) surface are

$$\frac{d^2r}{dt^2} = \frac{2m}{r(r-2m)}\left(\frac{dr}{dt}\right)^2 + (r-3m)\left(\frac{d\theta}{dt}\right)^2$$

$$\frac{d^2\theta}{dt^2} = -2\frac{(r-3m)}{r(r-2m)}\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right) \qquad (2)$$

These equations of motion describe the paths of light rays in a spherically symmetrical gravitational field. The figure below shows the paths of a set of parallel incoming rays.



The dotted circles indicate radii of m, 2m, ..., 6m from the mass center. Needless to say, a typical star's physical radius is much greater than it's gravitational radius m, so we will not find such severe deflection of light rays, even for rays grazing the surface of the star. However, for a "black hole" we can theoretically have rays of light passing at values of r on the same order of magnitude as m, resulting in the paths shown in this figure. Interestingly, a significant fraction of the oblique incoming rays are "scattered" back out, with a loop at r = 3m, which is the "light radius". As a consequence, if we shine a broad light on a black hole, we would expect to see a "halo" of back-scattered light outlining a circle with a radius of 3m.

To quantitatively assess the angular deflection of a ray of light passing near a large gravitating body, note that in terms of the variable u = dθ/dt the second geodesic equation (2) has the form (1/u)du = –[(2/r)(r–3m)/(r–2m)]dr, which can be integrated immediately to give ln(u) = ln(r–2m) – 3ln(r) + C, so we have

$$\frac{d\theta}{dt} = K\left(\frac{r-2m}{r^3}\right)$$

To determine the value of K, we divide the metric equation (1) by $(dt)^2$ and evaluate it at the perihelion $r = r_0$, where dr/dt = 0. This gives

$$1 = \frac{r_0^{\ 3}}{r_0 - 2m}\left(\frac{d\theta}{dt}\right)^2$$

Substituting into the previous equation we find $K^2 = r_0^{\ 3}/(r_0 - 2m)$, so we have

$$\frac{d\theta}{dt} = \sqrt{\frac{r_0^3}{r_0 - 2m}}\left(\frac{r-2m}{r^3}\right)$$

Now we can substitute this into the metric equation divided by $(dt)^2$ and solve for dr/dt to give

$$\frac{dr}{dt} = \frac{r-2m}{r}\sqrt{1-\left(\frac{r_0}{r}\right)^3\frac{r-2m}{r_0-2m}}$$

Dividing dθ/dt by dr/dt then gives

$$\frac{d\theta}{dr} = \frac{\sqrt{\frac{r_0^3}{r_0-2m}}}{\sqrt{1-\left(\frac{r_0}{r}\right)^3\frac{r-2m}{r_0-2m}}}\frac{1}{r^2}$$

Integrating this from $r = r_0$ to ∞ gives the mass-centered angle swept out by a photon as it moves from the perihelion out to an infinite distance. If we define $\rho = r_0/r$ the above equation can be written in the form

$$d\theta = \int_{\rho=0}^{1}\frac{1}{\sqrt{1-\rho^2}}\frac{1}{\sqrt{1-2\left(\frac{1-\rho^3}{1-\rho^2}\right)\frac{m}{r_0}}}\,d\rho$$

The magnitude of the second term in the right-hand square root is always less than 1 provided r_0 is greater than 3m (which is the radius of light-like circular orbits, as discussed further in Section 6.5), so we can expand the square root into a power series in that quantity. The result is

$$d\theta = \int_{\rho=0}^{1}\frac{1}{\sqrt{1-\rho^2}}\left(1+\frac{1}{2}\left[2\frac{1-\rho^3}{1-\rho^2}\frac{m}{r_0}\right]+\frac{3}{8}\left[2\frac{1-\rho^3}{1-\rho^2}\frac{m}{r_0}\right]^2+\frac{5}{16}\left[2\frac{1-\rho^3}{1-\rho^2}\frac{m}{r_0}\right]^3+\cdots\right)d\rho$$

This can be analytically integrated term by term. The integral of the first term is just π/2, as we would expect, since with a mass of m = 0 the photon would travel in a straight line, sweeping out a right angle as it moves from the perihelion to infinity. The remaining terms supply the "excess angle", which represents the angular deflection of the light ray. If m/r_0 is small, only the first-order term is significant. Of course, the path of light is symmetrical about the perihelion, so the total angular deflection between the asymptotes of the incoming and outgoing rays is *twice* the excess of the above integral beyond π/2. Focusing on just the first order term, the deflection is therefore

$$\delta = 2\frac{m}{r_0}\int_{\rho=0}^{1}\frac{1-\rho^3}{\left(1-\rho^2\right)^{3/2}}\,d\rho$$

Evaluating the integral

$$\int\frac{1-\rho^3}{\left(1-\rho^2\right)^{3/2}}\,d\rho = -\sqrt{\frac{1-\rho}{1+\rho}} - \sqrt{(1-\rho)(1+\rho)}$$

from $\rho = 0$ to 1 gives the constant factor 2, so the first-order deflection is $\delta = 4m/r_0$. This gives the relativistic value of 1.75 seconds of arc, which is twice the Newtonian value. To higher orders in m/r_0 we have

$$\delta = 4\left(\frac{m}{r_0}\right) + \left(\frac{15}{4}\pi - 4\right)\left(\frac{m}{r_0}\right)^2 + \left(-\frac{15}{2}\pi + \frac{122}{3}\right)\left(\frac{m}{r_0}\right)^3 + \cdots$$

The difficulty of performing precise measurement of optical starlight deflection during an eclipse can be gathered from the following list of results:

Optical Deflection of Starlight During Eclipses

Date	Location	arc secs
29 May 1919	Sobral	1.98 ± 0.16
	Principe	1.16 ± 0.40
21 Sep 1922	Australia	1.77 ± 0.40
		1.42 to 2.16
		1.72 ± 0.15
		1.82 ± 0.20
9 May 1929	Sumatra	2.24 ± 0.10
19 June 1936	USSR	2.73 ± 0.31
	Japan	1.28 to 2.13
20 May 1947	Brazil	2.01 ± 0.27
25 Feb 1952	Sudan	1.70 ± 0.10
30 Jun 1973	Mauritania	1.66 ± 0.19

Fortunately, much more accurate measurements can now be made in the radio wavelengths, especially of quasars, since such measurements can be made from observatories with the best equipment and careful preparation (rather than hurriedly in a remote location during a total eclipse). In particular, the use of Very Long Baseline Interferometry (VBLI), combining signals from widely separate observatories, gives a tremendous improvement in resolving power. With these techniques it's now possible to precisely measure the deflection (due to the Sun's gravitational field) of electromagnetic waves from stars at great angular distances from the Sun. By 1991 the observations of radio waves from stars consistently showed that the ratio of observed deflections to the deflections predicted by general relativity is 1.0001 ± 0.00001. Thus the dramatic announcement of 1919 has been retro-actively justified.

The first news of the results of Eddington's expedition reached Einstein by way of Lorentz, who on September 22 sent the telegram quoted at the beginning of this chapter. On the 7th of October Lorentz followed with a letter, providing details of Eddington's presentation to the "British Association at Bournemouth". Oddly enough, at this meeting Eddington reported that "one can say with certainty that the effect (at the solar limb) lies between 0.87" and 1.74", although he qualified this by saying the plates had been measured only preliminarily, and the final value was still to be determined. In any case, Lorentz's letter also included a rough analysis of the amount of deflection that would be expected due to ordinary refraction in the gas surrounding the Sun. His calculations indicated that a suitably chosen gas density at the Sun's surface could indeed produce a deflection on the order of 1", but for any realistic density profile the effect would drop off very rapidly for rays passing just slightly further from the Sun. Thus the effect of refraction, if there was any, would be easily distinguishable from the relativistic effect. He concluded

> We may surely believe (in view of the magnitude of the detected deflection) that, in reality, refraction is not involved at all, and your effect alone has been observed. This is certainly one of the finest results that science has ever accomplished, and we may be very pleased about it.

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Schwarzschild Black Hole

A black hole with zero charge $Q = 0$ and no angular momentum $J = 0$. The exteri solution for such a black hole is known as the Schwarzschild solution (or Schwarzschild metric), and is an exact unique solution to the Einstein field equati of general relativity for the general static isotropic metric (i.e., the most general metric tensor that can represent a static isotropic gravitational field),

$$d\tau^2 = B(r)\,dt^2 - A(r)\,dr^2 - r^2\,d\theta^2 - r^2\sin^2\theta\,d\phi^2.$$

In 1915, when Einstein first proposed them, the Einstein field equations appea so complicated that he did not believe that a solution would ever be found. He wa therefore quite surprised when, only a year later, Karl Schwarzschild (1916) discovered one by making the assumption of spherical symmetry.

In empty space, the Einstein field equations become

$$R_{\mu\nu} = 0,$$

where $R_{\mu\nu}$ is the Ricci curvature tensor. Reading off R_{rr}, $R_{\theta\theta}$, $R_{\phi\phi}$, and from the static isotropic metric (1) gives

$$R_{\phi\phi} = \sin^2\theta R_{\theta\theta},$$

so $R_{\phi\phi} = 0$ if $R_{\theta\theta} = 0$. Also

$$\frac{R_{rr}}{A} + \frac{R_{tt}}{B} = -\frac{1}{rA}\left(\frac{A'}{A} + \frac{B'}{B}\right) = 0,$$

so

$$\frac{A'}{A} = -\frac{B'}{B}$$

$$\frac{dA}{A} = -\frac{dB}{B}$$

$$\ln A = -\ln B + \ln[\text{const}]$$

$$\ln(AB) = \ln[\text{const}]$$

$$A(r)B(r) = [\text{const}]$$

But as $r \to \infty$, the metric tenor approaches the Minkowski metric, so

$$\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = 1$$

$$A(r) = \frac{1}{B(r)}$$

Plugging this into $R_{\theta\theta}$ and R_{rr} gives

$$R_{\theta\theta} = -1 + B'r + B = 0$$

$$R_{rr} = \frac{B''}{2B} + \frac{B'}{rB} = \frac{R'_{\theta\theta}}{2rB}$$

So we only have to make $R_{\theta\theta} = 0$, then $R_{rr} = 0$ and by (4), $R_{tt} = 0$.

$$\frac{d}{dr}[rB(r)] = rB'(r) + B(r) = 1,$$

so

$$rB(r) = r + [\text{const}].$$

Now, at great distance,

$$g_{tt} = -B \to -1 - 2\phi,$$

where the gravitational potential is

$$\phi = -\frac{MG}{c^2 r}.$$

Here, *M* is the mass of the black hole, *G* is the gravitational constant, and *c* is the speed of light. Note that it is very common to omit all factors of *c* (or equivalently set *c* = 1) in the equations of general relativity. Although slightly confusing, the *c* convention allows equations to be written more concisely, and no information is actually lost since the missing factors of *c* can always be unambiguously inserted dimensional analysis. Combining (16) and (17) gives the constant in (15) as $-2MG/c^2$, so

$$B(r) = 1 - \frac{2MG}{c^2 r}$$

$$A(r) = \left(1 - \frac{2MG}{c^2 r}\right)^{-1},$$

and the metric in standard form is therefore

$$d\tau^2 = \left(1 - \frac{2MG}{c^2 r}\right) dt^2 - \left(1 - \frac{2MG}{c^2 r}\right)^{-1} dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2.$$

This is the Schwarzschild solution in standard form.

The radius at which the metric becomes singular is

$$r_S = \frac{2MG}{c^2},$$

known as the Schwarzschild radius.

The Killing vector fields for the Schwarzschild solution are $\partial/\partial t$, $x\partial/\partial y - y\partial/\partial x$, $y\partial/\partial z - z\partial/\partial y$, and $z\partial/\partial x - x\partial/\partial z$.

An exact solution turns out to also be possible for a spherical body with constant density; see Schwarzschild black hole--constant density.

SEE ALSO: Birkhoff's Theorem, Black Hole, Eddington-Robertson Metric, Einstein Fi Equations, General Relativity, Schwarzschild Black Hole--Constant Density, Schwarzschild Black Hole--Eddington-Finkelstein Coordinates, Schwarzschild Blac Hole--Isotropic Form, Schwarzschild Black Hole--Kruskal Coordinates, Schwarzsch Black Hole--Radial Infall, Schwarzschild Radius, Static Isotropic Metric

REFERENCES:

Schwarzschild, K. "Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften* **1**, 189-196, 1916.

Schwarzschild, K. *Sitzungsber. Deutsch. Akad. Wissenschaft Berlin, Kl. Math. Phys. Tech.*, 189-19 1916.

Shapiro, S. L. and Teukolsky, S. A. "Schwarzschild Black Holes." §12.3 in *Black Holes, White Dwa and Neutron Stars: The Physics of Compact Objects.* New York: Wiley, pp. 338-339, 1983.

Weinberg, S. "The Schwarzschild Solution." §8.2 in *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity.* New York: Wiley, pp. 179-182, 1972.