1)

Notes 56(2)

Aharonov Bohm Effects and Parallel Transport

Parallel transport is defined for the arbitrary tensor T by:

$$\frac{dT}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial T}{\partial x^\mu} = 0. \quad -(1)$$

This result comes from the rule for differentiation of a function of a function, i.e. if:

$$T = T(x^\mu(\lambda)), \quad -(2)$$

$$\frac{dT}{d\lambda} = \frac{dT}{dx^\mu}\frac{dx^\mu}{d\lambda}. \quad -(3)$$

The rule (3) may be extended to the covariant derivative of a tensor T of any rank:

$$\frac{DT}{d\lambda} = \frac{dx^\mu}{d\lambda} D_\mu T \quad -(4)$$

where $T = T(x^\mu(\lambda))$. In these notes we extend the rule (4) to the exterior covariant derivative of Cartan, defined by the first Cartan structure equation:

$$T^a = D \wedge q^a = d \wedge q^a + \omega^a{}_b \wedge q^b \quad -(5)$$

and second Cartan structure equation:

$$R^a{}_b = D \wedge \omega^a{}_b = d \wedge \omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \quad -(6)$$

2) It will be proven in these notes that:

$$\frac{dx^\mu}{d\lambda}\left(D \wedge A\right)_{\mu\nu} = \frac{dx^\mu}{d\lambda} F^a_{\mu\nu} \quad -(7)$$

is the origin of the Lorentz force equation and of the class of Aharonov Bohm effects. Here $F^a_{\mu\nu}$ is the e/m field tensor:

$$F^a = D \wedge A^a. \quad -(8)$$

Now consider:

$$A^a_\mu = A^a_\mu\left(x^\nu(\lambda)\right) \quad -(9)$$

and

$$A^a_\nu = A^a_\nu\left(x^\mu(\lambda)\right) \quad -(10)$$

with:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \omega^a_{\mu b} A^b_\nu - \omega^a_{\nu b} A^b_\mu. \quad -(11)$$

Then:

$$\frac{DA^a_\mu}{d\lambda} = \frac{dx^\nu}{d\lambda} D_\nu A^a_\mu \quad -(12)$$

$$\frac{DA^a_\nu}{d\lambda} = \frac{dx^\mu}{d\lambda} D_\mu A^a_\nu. \quad -(13)$$

Using the tetrad postulate it follows that:

$$\boxed{\frac{DA^a_\mu}{d\lambda} = \frac{DA^a_\nu}{d\lambda} = 0}. \quad -(14)$$

3) This means that $A^a_\mu = A^{(0)}q^a_\mu$ is parallel transported along any curve $x^\mu(\lambda)$. It follows that:

$$\frac{dx^\mu}{d\lambda}\left(D_\nu A^a_\mu - D_\mu A^a_\nu\right) = 0. \quad -(15)$$

The exterior covariant derivative is defined as:

$$\begin{aligned}(D \wedge A)^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \omega^a_{\mu b} A^b_\nu - \omega^a_{\nu b} A^b_\mu \\ &= F^a_{\mu\nu} \quad -(16)\end{aligned}$$

where $F^a_{\mu\nu}$ is the electromagnetic field tensor.

Now use the result: $\quad -(17)$

$$\boxed{\begin{aligned}&D_\mu A^a_\nu - D_\nu A^a_\mu \\ &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \omega^a_{\mu b} A^b_\nu - \omega^a_{\nu b} A^b_\mu - F^a_{\mu\nu} \\ &= 0.\end{aligned}}$$

It follows from eqns (15) and (17) that the Cartan structure equation is parallel transported along any curve:

$$\boxed{\frac{dx^\mu}{d\lambda}\left(F^a - D \wedge A^a\right)_{\mu\nu} = 0} \quad -(18)$$

and so:

4)

$$\boxed{\frac{dx^\mu}{d\lambda}\left(D\wedge A^a\right)_{\mu\nu} = \frac{dx^\mu}{d\lambda} F^a_{\mu\nu}} \quad -(19)$$

This is the origin of the Lorentz force equation in ECE theory. In the absence of acceleration

$$\frac{dx^\mu}{d\lambda} F^a_{\mu\nu} = \frac{dx^\mu}{d\lambda}\left(D\wedge A^a\right)_{\mu\nu} = 0. \quad -(20)$$

In the absence of acceleration the exterior covariant derivative is parallel transported.

Similarly:

$$\boxed{\frac{dx^\mu}{d\lambda} R^a{}_{b\mu\nu} = \frac{dx^\mu}{d\lambda}\left(D\wedge \omega^a{}_b\right)_{\mu\nu}} \quad -(21)$$

In the absence of force, the exterior covariant derivative of the spin connection is parallel transported:

$$\frac{dx^\mu}{d\lambda} R^a{}_{b\mu\nu} = \frac{dx^\mu}{d\lambda}\left(D\wedge \omega^a{}_b\right)_{\mu\nu} = 0 \quad -(22).$$

The class of Aharonov Bohm effects may now be defined in terms of these equations.

5) The Aharonov Bohm effects are defined by eqn. (20). When there is no field there may still be a potential defined by:

$$A^{(0)}\left(d \wedge q^a + \omega^a{}_b \wedge q^b\right) = 0 \quad -(23)$$

for all $A^{(0)}$ and all paths $x^\mu(\lambda)$. The geometry of the AB effects is always defined by:

$$\boxed{D \wedge q^a = d \wedge q^a + \omega^a{}_b \wedge q^b = 0} \quad -(24)$$

for the electromagnetic field or by:

$$\boxed{D \wedge \omega^a{}_b = d \wedge \omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = 0} \quad -(25)$$

for the gravitational field.

The Aharonov Bohm effects are therefore due to parallel transport of the exterior covariant derivative.

The AB effects are therefore effects of generally covariant unified field theory, of spinning and curving spacetime through the spin connection. In the standard model there are no AB effects because $F = d \wedge A$, and if F is zero so is A.