Notes for Paper 55, Part One

Derivation of the Euler Equations from ECE

Define the angular momentum as:

$$J^a_\mu = J^{(0)} q^a_\mu \quad -(1)$$

(pp 287 ff of volume one). Now use the Cartan structure equation:

$$T^a = d \wedge q^a + \omega^a{}_b \wedge q^b \quad -(2)$$

to find the torque:

$$\boxed{N^a = c\left(d \wedge J^a + \omega^a{}_b \wedge J^b\right)} \quad -(3)$$

This is the generally covariant Euler equation of motion, more accurately the g. c. rotational equation.

In the classical limit the Euler equation of motion is:

$$\underline{N} = \left(\frac{d\underline{J}}{dt}\right)_{\text{lab frame}}$$

$$= \left(\frac{d\underline{J}}{dt}\right)_{\text{moving frame}} + \underline{\omega} \times \underline{J}$$

2) As described by Marion and Thornton on pages 388 ff:

$$\underline{N} = \left(\frac{d\underline{L}}{dt} \right)_{fixed} \quad -(5)$$

is valid in an inertial frame only.

Eqn. (3) is automatically relativistic, and shows that:

$$\boxed{N^a = c J^{(0)} T^a}, \quad -(6)$$

relating the torque form N^a and the Cartan torsion T^a. The Cartan structure equation defines the torque in general relativity. The angular momentum is a type of tetrad field, a first inferred in rel. ore. In eq. (4) $\underline{\omega}$ is the angular velocity in the lab frame in classical dynamics.

Eq (3) is closely analogous to the equation defining the electromagnetic field:

$$F^a = d \wedge A^a + \omega^a{}_b \wedge A^b$$

(7)

3)

So the $\omega^a{}_b \wedge J^b$ term originates in a spinning frame of reference, i.e. spinning spacetime itself. If spacetime is not spinning there is no Cartan torsion, so there is no torque:

$$N^a = 0. \quad -(6)$$

This situation describes Newtonian dynamics, which is the weak field limit of a Riemann geometry which has curvature, $R^a{}_b$, <u>but no torsion</u>, T^a. In Newtonian dynamics the torque is well known to be:

$$\underline{N} = \underline{r} \times \underline{F} = \dot{\underline{J}}, \quad -(7)$$

where $\underline{F}$ is force and $\underline{r}$ is arm length. The Newtonian force $\underline{F}$ originates in the Riemann curvature and not in the Cartan torsion.

So in classical dynamics there is a mixture of concepts and an internal inconsistency, classical dynamics attempts to describe torque without torsion. The self-consistent and generally covariant description of rotational

4) dynamics is eq. (3). Therefore there are rotational effects in ECE that do not exist in classical dynamics. From eq. (6) it is found that torque is the Cartan torsion within a factor $cJ^{(0)}$. So torque interacts with a gravitational field through the first Bianchi identity:

$$d \wedge N^a + \omega^a{}_b \wedge N^b$$
$$= c J^{(0)} R^a{}_b \wedge q^b$$
$$= c R^a{}_b \wedge J^b, \quad -(8).$$

i.e.

$$\boxed{D \wedge N^a = c R^a{}_b \wedge J^b} \quad -(9)$$

In the absence of a gravitational field:

$$R^a{}_b = 0 \quad -(10)$$

so:

$$D \wedge N^a = 0. \quad -(11)$$

The Newtonian torque (7) is calculated from:

$$D \wedge R^a{}_b = 0. \quad -(12)$$