1)

Notes for Paper 54, Part 6

Non-Linear Terms in the Action

The complete derivation of the scalar curvature R in terms of the action S is as follows. Start with:

$$q_\mu^a = \exp(iS/\hbar)\, q_\mu^a(0) \quad -(1)$$

Differentiate:

$$\partial^\nu q_\mu^a = \partial^\nu\left(e^{iS/\hbar} q_\mu^a(0)\right)$$

$$= \frac{i}{\hbar}\partial^\nu S\, e^{iS/\hbar} q_\mu^a(0)$$

$$\boxed{\partial^\nu q_\mu^a = \frac{i}{\hbar}\left(\partial^\nu S\right) q_\mu^a} \quad -(2)$$

This is the generally covariant Schrödinger equation.

Differentiate eq (2):

$$\partial_\nu\left(\partial^\nu q_\mu^a\right) = \frac{i}{\hbar}\partial_\nu\left(\left(\partial^\nu S\right) q_\mu^a\right), \quad -(3)$$

i.e.

$$\Box q_\mu^a = \frac{i}{\hbar}\left(\Box S + \frac{i}{\hbar}\partial^\nu S\, \partial_\nu S\right) q_\mu^a$$

$$= R\, q_\mu^a \quad -(4)$$

$$\boxed{R = \frac{i}{\hbar}\left(\Box S + \frac{i}{\hbar}\partial^\nu S\, \partial_\nu S\right)} \quad -(5)$$

2)

Therefore the derivation in notes part S assumes

$$\frac{1}{\hbar} \partial^\mu S \, \partial_\mu S \ll \Box S . \quad -(6)$$

This may be a good approximation but more generally eq (5) holds.

Even more generally $\hbar$ in eq (1) may be replaced by an action S_0, so:

$$q_\mu^a = \exp(i S / S_0) \, q_\mu^a(0) \quad -(7)$$

and:

$$R = \frac{i}{S_0} \left(\Box S + \frac{i}{S_0} \partial^\mu S \, \partial_\mu S \right) . \quad -(8)$$

So we have not assumed that S_0 is $\hbar$, the theory has been kept as general as possible. In the limit of the Dirac equation:

$$R \rightarrow \frac{m^2 c^2}{\hbar^2} . \quad -(9)$$

This identifies the Planck constant as:

$$\boxed{\frac{m^2 c^2}{\hbar^2} = \frac{i}{S_0} \left(\Box S + \frac{i}{S_0} \partial^\mu S \, \partial_\mu S \right)}$$

where: