37(2): Relativistic Developments

Consider the Einstein energy equation of special relativity:

$$E^2 = p^2c^2 + m^2c^4 \quad (1)$$

where E is the total relativistic energy, p is the relativistic momentum, m the particle mass and c the speed of light in vacuo. Eq. (1) is quantized using the same rules as in non-relativistic quantum mechanics:

$$E\psi = i\hbar\frac{\partial\psi}{\partial t} \quad (2)$$

and

$$p\psi = -i\hbar\nabla\psi \quad (3)$$

where ψ is the total wave function. Eq. (1) can be reduced to the Schroedinger equation as follows. Write it as:

$$(E - mc^2)(E + mc^2) = p^2c^2 \quad (4)$$

So

$$E - mc^2 = \frac{p^2c^2}{E + mc^2} \quad (5)$$

In this equation:

$$H = E + U \quad (6)$$

where U is the potential energy, so:

$$H - U - mc^2 = \frac{p^2c^2}{E + mc^2} \quad (7)$$

Define:

$$H_0 := H - mc^2 \quad (8)$$

so

$$H_0 = \frac{p^2c^2}{E + mc^2} + U \quad (9)$$

In general:

$$E = \gamma mc^2 \quad (10)$$

2) where $$\gamma = \left(1 - \frac{v_N^2}{c^2}\right)^{-1/2} \quad -(11)$$

is the Lorentz factor.

In the limit:

$$v_N << c \quad -(12)$$

it follows that $$\gamma \rightarrow 1 \quad -(13)$$

and $$H_0 = \frac{p^2 c^2}{2mc^2} + U \quad -(14)$$

i.e. $$H_0 = \frac{p^2}{2m} + U \quad -(15)$$

which is the classical result, Q.E.D. Eq. (15)

quantizes to: $$H_0 \phi = -\frac{\nabla^2 \phi}{2m} + U\phi \quad -(16)$$

$$= E\phi = i\hbar \frac{\partial \phi}{\partial t}$$

This is the Schroedinger equation:

$$-\frac{\hbar^2}{2m} \nabla^2 \phi + U\phi = i\hbar \frac{\partial \phi}{\partial t} \quad -(17)$$

If $$\phi = \phi_1(r)\phi_2(t) \quad -(18)$$

eq. (17) splits into:

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \phi_1}{\partial r^2} + U(r)\phi_1 = E\phi_1 \quad -(19)$$

and $$i\hbar \frac{\partial \phi_2}{\partial t} = E\phi_2 \quad -(20)$$

3) The solution of eq. (20) is:

$$\psi_2 = \exp\left(-\frac{iEt}{\hbar}\right) \quad -(21)$$

In the non-relativistic Schroedinger limit:

$$E_n = -\frac{\mu e^4}{32\pi^2 \epsilon_0^2 \hbar^2 n^2} \quad -(22)$$

where μ is the reduced mass, e is the charge on the proton, ϵ_0 is the permittivity of the vacuum, $\hbar$ the reduced Planck constant, and n the principal quantum number.

These results have been obtained by using the non-relativistic approximation (12). The rigorously correct energy levels of the Dirac H atom are:

$$E_{nJ} = E_n\left(1 + \left(\frac{\alpha}{n}\right)^2\left(\frac{n}{J+\frac{1}{2}} - \frac{3}{4}\right)\right) \quad -(23)$$

where α is the fine structure constant and J is the total angular momentum quantum number:

$$J = L+S, \ldots, L-S \quad -(24)$$

and

$$L = 0, \ldots, n-1 \quad -(25)$$

The spin quantum number takes the values:

$$S = \pm\frac{1}{2} \quad -(26)$$

The quantum numbers J and S are missing from the Schroedinger H atom because the above derivation of the Schroedinger equation from the relativistic energy equation (1) does not use the SU(2) basis.

In the Dirac H atom, the energy levels of $2P_{1/2}$

) and $^2S_{1/2}$ are the same because J is the same in
both:

$^2S_{1/2}$: $n = 1,\ L = 0,\ S = 1/2,\ J = 1/2$ — (27)

$^2P_{1/2}$: $n = 1,\ L = 1,\ S = -1/2,\ J = 1/2$ — (28)

Therefore there is no Lamb shift in the Dirac H atom.