437(1): Development of the Separation of Variables Method

The usual separation of variables method is applied to the Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial r^2} + U(r)\psi = i\hbar\frac{\partial\psi}{\partial t} \quad -(1)$$

using:

$$\psi = \psi_1(r)\psi_2(t), \quad -(2)$$

where by definition:

$$\frac{\partial\psi_1(r)}{\partial t} = 0 \quad -(3)$$

and

$$\frac{\partial\psi_2(t)}{\partial r} = 0. \quad -(4)$$

It follows that:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2}(\psi_1\psi_2) + U(r)\psi_1\psi_2 = i\hbar\frac{\partial}{\partial t}(\psi_1\psi_2) \quad -(5)$$

In this equation:

$$\frac{\partial^2}{\partial r^2}(\psi_1\psi_2) = \psi_1\frac{\partial^2\psi_2}{\partial r^2} + 2\frac{\partial\psi_1}{\partial r}\frac{\partial\psi_2}{\partial r} + \psi_2\frac{\partial^2\psi_1}{\partial r^2} \quad -(6)$$

$$= \psi_2\frac{\partial^2\psi_1}{\partial r^2}$$

using eq. (4).

Also:

$$\frac{\partial}{\partial t}(\psi_1\psi_2) = \psi_1\frac{\partial\psi_2}{\partial t} + \psi_2\frac{\partial\psi_1}{\partial t} \quad -(7)$$

$$= \psi_1\frac{\partial\psi_2}{\partial t}$$

using eq. (3). So eq. (5) reduces to:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi_1}{\partial r^2}\psi_2 + U(r)\psi_1\psi_2 = i\hbar\psi_1\frac{\partial\psi_2}{\partial t} \quad -(8)$$

2) Divide through by $\psi_1 \psi_2$:

$$-\frac{\hbar^2}{2m}\frac{1}{\psi_1}\frac{\partial^2 \psi_1}{\partial r^2} + U(r) = \frac{i\hbar}{\psi_2}\frac{\partial \psi_2}{\partial t} \quad -(9)$$

This divides into two equations:

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi_1}{\partial r^2} + U(r)\psi_1 = E\psi_1 \quad -(10)$$

and

$$i\hbar \frac{\partial \psi_2}{\partial t} = E\psi_2 \quad -(11)$$

These are the time independent and time dependent Schroedinger equations.

The solution of eqn. (11) is:

$$\psi_2(t) = \exp\left(-i\frac{E}{\hbar}t\right) \quad -(12)$$

so

$$\boxed{\psi = \exp\left(-i\frac{E}{\hbar}t\right)\psi_1(r)} \quad -(13)$$

Eq. (13) is the non relativistic limit of the Minkowski metric:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 \quad -(14)$$

At this point, at the end of the calculation, Eqs. (10) and (11) are transformed into n space

) defined by:

$$ds^2 = c^2 d\tau^2 = m(r)c^2 dt^2 - \frac{dr^2}{m(r)} - r^2 d\phi^2 \quad -(15)$$

In a stationary metric, $m(r)$ has no time dependence. The transformation is accomplished using the quantization rules:

$$t \rightarrow m(r)^{1/2} t \quad -(16)$$

and

$$r \rightarrow \frac{r}{m(r)^{1/2}} \quad -(17)$$

So:

$$\boxed{\psi_2(t) = \exp\left(-\frac{iE}{\hbar} m(r)^{1/2} t\right)} \quad -(18.)$$

and

$$\psi_1(r) \rightarrow \psi_1\left(\frac{r}{m(r)^{1/2}}\right) \quad -(19)$$

in eq. (10).

By definition of a stationary metric (Carroll, online notes):

$$\frac{dm(r)}{dt} = 0 \quad -(20)$$

From eq. (11):

$$\frac{\partial \psi_2(t)}{\partial t} = -\frac{iE}{\hbar} m^{1/2}(r) \psi_2(t) \quad -(21)$$

so:

$$\langle E \rangle = E \int \psi_2^* m^{1/2}(r) \psi_2 \, d\tau \quad -(22)$$

+) and

$$\langle E \rangle = E \int m^{1/2}(r)\, d\tau \quad -(23)$$

As in Note 436(4), the rigorously correct expectation value in Eq. (23) is:

$$\langle E \rangle = E \frac{\int \psi_2^* m^{1/2}(r)\, \psi_2\, d\tau}{\int \psi_2^* \psi_2\, d\tau} \quad -(24)$$

In the familiar case:

$$m(r) = 1 \quad -(25)$$

it is clear that:

$$\langle E \rangle = E \quad -(26)$$

However when

$$m(r) \neq 1 \quad -(27)$$

then:

$$\boxed{\langle E \rangle = E \frac{\int_0^r 4\pi\, m(r)^{1/2}\, r^2\, dr}{\int_0^r 4\pi r^2\, dr}} \quad -(28)$$

i.e.

$$\langle E \rangle = E \left(\frac{\int_0^r 4\pi\, m(r)^{1/2}\, r^2\, dr}{\frac{4}{3}\pi r^3} \right) \quad -(29)$$

The Born interpretation means that:

$$\int \psi_2^* \psi_2\, d\tau = 1 \quad -(30)$$

) So the correctly normalized wavefunction is:

$$\boxed{\psi_2 = \frac{\exp\left(-i\frac{E}{\hbar} m(r)^{1/2} t\right)}{V_0^{1/2}}} \quad -(31)$$

here

$$V_0 = \frac{4}{3}\pi r^3 \quad -(32)$$

For example if:

$$m(r) = 1 - \frac{r_0}{r} \quad -(33)$$

then

$$m(r)^{1/2} \sim 1 - \frac{r_0}{2r} \quad -(34)$$

and

$$\langle E \rangle = E\left(1 - \frac{r_0}{2}\frac{\int_0^r 4\pi r\, dr}{\frac{4}{3}\pi r^3}\right) \quad -(35)$$

$$= E\left(1 - \frac{3}{4}\frac{r_0}{r}\right)$$

$$\xrightarrow[r\to\infty]{} E$$

and the correct result is obtained in flat space $(r \to \infty)$.