435(6): Rules for Quantization in m Space

1) Transform the wave function using

$$r \to \frac{r}{m(r)^{1/2}}, \quad t \to m(r)^{1/2} t \quad -(1)$$

2) Quantize the energy and momentum using:

$$E\psi = i\hbar \frac{\partial\psi}{\partial t} \quad -(2)$$

$$\underline{p}\psi = -i\hbar \underline{\nabla}\psi \quad -(3)$$

Example: Free Particle

The free particle wave function:

$$\psi(t, r) = \exp(-i(\omega t - \kappa r)) \quad -(4)$$

is transformed to:

$$\psi(t, r) = \exp\left(-i\left(m^{1/2}(r)\omega t - \frac{\kappa r}{m(r)^{1/2}}\right)\right)$$

and the time dependent Schroedinger equation is: $-(5)$

$$i\hbar \frac{\partial\psi}{\partial t} = H\psi \quad -(6)$$

The Hamiltonian is transformed to:

$$H = \frac{p^2}{2m\,m(r)} \quad -(7)$$

It follows that:

$$E = i\hbar \int \psi^* \frac{\partial \psi}{\partial t} d\tau = -\frac{\hbar^2}{2m} \int \psi^* \nabla^2 \psi \, d\tau \quad -(8)$$

in which the wave function is given by Eq. (5).

From Eq. (5):

$$\frac{\partial \psi}{\partial t}(r,t) = -i m^{1/2}(r)\, \omega \, \psi(r,t) \quad -(9)$$

So

$$\boxed{E = \hbar\omega \int \psi^* m^{1/2}(r) \psi \, d\tau} \quad -(10)$$

In the limit:

$$m(r) \rightarrow 1 \quad -(11)$$

the Planck quantization is obtained:

$$E = \hbar\omega \quad -(12)$$

because:

$$\int \psi^* \psi \, d\tau = 1. \quad -(13)$$

Therefore for the free particle:

$$E = \hbar\omega \int \psi^* m^{1/2}(r) \psi \, d\tau = -\frac{\hbar^2}{2m} \int \psi^* \nabla^2 \psi \, d\tau \quad -(14)$$

For a radial wave function:

$$E = \hbar\omega \int \psi^* m^{1/2}(r) \psi \, d\tau = -\frac{\hbar^2}{2m} \int \psi^* \frac{\partial^2}{\partial r^2} \left(\frac{ikr}{m(r)^{1/2}} \right) d\tau \quad -(15)$$

In the limit:

$$m(r) \rightarrow 1 \quad -(16)$$

3)

$$E = \hbar\omega = \frac{\hbar^2 \kappa^2}{2m} \quad - (16)$$

The energy levels are shifted by eq. (14). These are the free particle energy levels in the κ space, i.e. free particle energy levels in the presence of the vacuum.

This method of calculation initially transforms the wavefunction using eq. (1), and the Schrodinger quantization rules (2) and (3) are applied to the transformed wavefunction