433(6): Check on the Plane Wave Solution

Start with:

$$E^2 = c^2 p^2 + m^2 c^4 \quad - (1)$$

and quantize using:

$$p^\mu = i\hbar \partial^\mu \quad - (2)$$

i.e.

$$\left(\frac{E}{c}, \underline{p}\right) = i\hbar\left(\frac{1}{c}\frac{\partial}{\partial t}, -\underline{\nabla}\right) \quad - (3)$$

so

$$E\phi = i\hbar\frac{\partial\phi}{\partial t}, \quad \underline{p}\phi = -i\hbar\underline{\nabla}\phi \quad - (4)$$

From eqs. (1) and (4):

$$-\hbar^2\frac{\partial^2\phi}{\partial t^2} = -c^2\hbar^2\nabla^2\phi + m^2c^4\phi \quad - (5)$$

i.e.

$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2\right)\phi = 0 \quad - (6)$$

So the Einstein energy equation (1) quantizes into the d'Alembert wave equation (6), as is well known.

Now use the plane wave:

$$\phi = \exp(-i(\omega t - \kappa Z)) \quad - (7)$$

and it follows that:

$$\frac{\partial^2\phi}{\partial t^2} = -\omega^2\phi \; ; \; \frac{\partial^2\phi}{\partial Z^2} = -\kappa^2\phi \quad - (8)$$

so:

$$\Box\phi = \left(\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi\right) = -\left(\frac{\omega^2}{c^2} - \kappa^2\right)\phi \quad - (9)$$

i.e

$$\Box\phi + \left(\frac{\omega^2}{c^2} - \kappa^2\right)\phi = 0 \quad - (10)$$

From eqs. (6) and (10):

$$\frac{\omega^2}{c^2} - \kappa^2 = \left(\frac{mc}{\hbar}\right)^2 \quad - (11)$$

i.e.

$$\hbar^2\omega^2 = c^2\hbar^2\kappa^2 + m^2c^4 \quad - (12)$$

Finally, using:

$$E = \hbar\omega \quad -(13)$$

and

$$\underline{p} = \hbar\underline{\kappa} \quad -(14)$$

Eq. (12) becomes

$$E^2 = c^2p^2 + m^2c^4 \quad -(15)$$

which is eq. (1), Q.E.D.

So eq. (7) is a solution of eq. (6) provided that Eq. (1) is true. However, eq. (6) is derived from eq. (1) so eq. (7) is always a solution of eq. (6), Q.E.D.

Similarly eq. (7) is a solution of:

$$\left(\Box + n(r)\left(\frac{mc}{\hbar}\right)^2\right)\phi = 0 \quad -(16)$$

provided that

$$E^2 = c^2p^2 + n(r)m^2c^4 \quad -(17)$$

These calculations can be run through Maxima to triple check them.
