425 (?): Hamilton Equations in Special Relativity

In this case:

$$\mathcal{L} = \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - H \quad -(1)$$

$$= p\dot{q} - H$$

The relativistic lagrangian is:

$$\mathcal{L} = -\frac{mc^2}{\gamma} + \frac{mMG}{r}$$

and the relativistic hamiltonian is:

$$H = \gamma mc^2 - \frac{mMG}{r} \quad -(3)$$

so

$$p\dot{q} = -\frac{mc^2}{\gamma} + \gamma mc^2$$

$$= mc^2\left(\frac{\gamma^2-1}{\gamma}\right)$$

$$= \frac{m^2c^4(\gamma^2-1)}{\gamma mc^2}$$

$$= \frac{E^2 - m^2c^4}{E} = \frac{p^2c^2}{\gamma mc^2} \quad -(4)$$

This result is given by Marion and Thornton, third edition after eq. (14.114).

Now use:

$$p = p_r = \gamma m v_N \quad -(5)$$

and

$$\dot{q}_r = \dot{r} = v_N \quad -(6)$$

to find that

$$p\dot{q} = p_r\dot{q}_r = \gamma m v_N^2 = \frac{p^2c^2}{\gamma mc^2} \quad -(7)$$

Q.E.D.

2) So the canonically conjugate variable of the Hamiltonian dynamics are:

$$p_r = \gamma m v_N \quad -(8)$$

and

$$q_r = r \quad -(9)$$

The Hamilton equations are:

$$\dot{r} = \frac{\partial H}{\partial p_r} \quad -(10)$$

and

$$\dot{p}_r = -\frac{\partial H}{\partial r} \quad -(11)$$

where

$$H = mc^2\left(1-\frac{v_N^2}{c^2}\right)^{-1/2} - \frac{mMG}{r} \quad -(12)$$

From eq. (11)

$$\boxed{\frac{d}{dt}\left(\gamma m v_N\right) = -\frac{mMG}{r^2}} \quad -(13)$$

Eq. (10) is developed as in Note 425(2):

$$\dot{r} = \frac{\partial H}{\partial p} = \frac{\partial H}{\partial v_N}\frac{\partial v_N}{\partial p} \quad -(14)$$

where

$$p = \gamma m v_N \quad -(15)$$

$$= m\left(1-\frac{v_N^2}{c^2}\right)^{-1/2} v_N$$

so

$$\frac{\partial p}{\partial v_N} = m\left(\gamma + v_N\frac{d}{dv_N}\left(1-\frac{v_N^2}{c^2}\right)^{-1/2}\right)$$

$$= m\left(\gamma + \gamma^3\frac{v_N^2}{c^2}\right)$$

3)

$$= m\gamma^3\left(\frac{1}{\gamma^2}+\frac{v_N^2}{c^2}\right)$$

$$= \gamma^3 m \quad -(16)$$

Therefore:

$$\frac{\partial H}{\partial p} = \frac{1}{m\gamma^3}\frac{\partial H}{\partial v_N} \quad -(17)$$

where

$$\frac{\partial H}{\partial v_N} = m\gamma^3 v_N \quad -(18)$$

Therefore:

$$\frac{\partial H}{\partial p} = v_N = \dot{r} \quad -(19)$$

and

$$v_N = \dot{r} \quad -(20)$$

which is Eq. (6), Q.E.D.

<u>Summary</u>

The Hamilton equation:

$$\dot{p}_r = -\frac{\partial H}{\partial r} \quad -(21)$$

gives

$$\frac{d}{dt}(\gamma m v_N) = -\frac{mMG}{r^2} \quad -(22)$$

and the Hamilton equation:

$$\dot{r} = \frac{\partial H}{\partial p_r} \quad -(23)$$

gives

$$v_N = \dot{r} \quad -(24)$$

Now consider the Lagrangian analysis of previous UFT papers:

4)

$$\mathcal{L} = -mc^2\left(1 - \frac{\dot{r}^2 + r^2\dot{\phi}^2}{c^2}\right)^{1/2} + \frac{mMG}{r} \quad -(25)$$

with:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\partial \mathcal{L}}{\partial r} \quad -(26)$$

and

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\partial \mathcal{L}}{\partial \phi} \quad -(27)$$

Eq. (26) gives:

$$\frac{d}{dt}(\gamma m\dot{r}) - \gamma m r\dot{\phi}^2 = -\frac{mMG}{r^2} \quad -(28)$$

and Eq. (27) gives:

$$\frac{dL}{dt} = 0 \quad -(29)$$

where

$$L = \gamma m r^2 \dot{\phi} \quad -(30)$$

The lagrangian analysis introduces:

$$p_\phi = L, \quad q_\phi = \phi \quad -(31)$$

which are canonically conjugate variables of Hamiltonian dynamics that exist in addition to:

$$p_r = \gamma m v_N, \quad q_r = r \quad -(32)$$

The Hamiltonian equivalent to Eq. (25) is

$$H = mc^2\left(1 - \frac{\dot{r}^2 + r^2\dot{\phi}^2}{c^2}\right)^{-1/2} - \frac{mMG}{r} \quad -(33)$$

The Hamilton equation:

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} \quad -(34)$$

5) gives:

$$\dot{L} = \frac{dL}{dt} = 0 \quad -(35)$$

which is consistent with the fact that L defined by Eq. (30) is a constant of motion. So the Hamiltonian and Lagrangian analyses in special relativity are self consistent, Q.E.D.

The second Hamilton equation:

$$\dot{v}_\phi = \frac{\partial H}{\partial p_\phi} \quad -(36)$$

gives:

$$\dot{\phi} = \frac{\partial H}{\partial L} \quad -(37)$$

This is new information not given by Lagrangian dynamics. In eq. (37)

$$p_\phi = L = \gamma m v_\phi \quad -(38)$$

where

$$v_\phi = r\dot{\phi}. \quad -(39)$$

so for rotational dynamics:

$$L = m\left(1 - \frac{v_\phi^2}{c^2}\right)^{-1/2} v_\phi \quad -(40)$$

and

$$\dot{\phi} = \frac{\partial H}{\partial L} = \frac{\partial H}{\partial v_\phi}\frac{\partial v_\phi}{\partial L} \quad -(41)$$

From eq. (40):

$$\frac{\partial L}{\partial v_\phi} = r\gamma^3 m \quad -(42)$$

) where $$\gamma = \left(1 - \frac{v_\phi^2}{c^2}\right)^{-1/2} \quad -(43)$$

Therefore $$\frac{\partial H}{\partial p_\phi} = \frac{1}{m\gamma^3}\frac{\partial H}{\partial v_\phi} \quad -(44)$$

where $$\frac{\partial H}{\partial v_\phi} = m\gamma^3 v_\phi \quad -(45)$$

so $$\frac{\partial H}{\partial p_\phi} = \frac{\partial H}{\partial L} = \frac{\partial H}{\partial v_\phi}\frac{\partial v_\phi}{\partial L} = \dot{\phi} \quad -(46)$$

Q.E.D. Here: $$v_\phi = r\dot{\phi} = \frac{L}{mr} \quad -(47)$$

and $$L = \gamma m v_\phi r \quad -(48)$$

<u>Summary</u>

The canonically conjugate variables are:

$$p_r = \gamma m \dot{r}, \quad q_r = r \quad -(49)$$

and $$p_\phi = L, \quad q_\phi = \phi. \quad -(50)$$

The first Hamilton equation gives:

$$\frac{d}{dt}(\gamma m \dot{r}) = -\frac{mMG}{r^2} \quad -(51)$$

and the second Hamilton equation gives:

$$\dot{\phi} = \frac{\partial H}{\partial L}, \quad L = \gamma m r^2 \dot{\phi} \quad -(52)$$

The first Hamilton equation also gives:

7)

$$\frac{dL}{dt} = 0 \quad -(53)$$

$$\boxed{\dot{q} = \frac{\partial H}{\partial p}} \longrightarrow \boxed{\begin{aligned}\dot{q}_r &= \frac{\partial H}{\partial p_r} \text{ and} \\ \dot{q}_\phi &= \frac{\partial H}{\partial p_\phi}\end{aligned}} \quad -(54)$$

$$\boxed{\dot{p} = -\frac{\partial H}{\partial q}} \longrightarrow \boxed{\begin{aligned}\dot{p}_r &= -\frac{\partial H}{\partial q_r} \text{ and} \\ \dot{p}_\phi &= -\frac{\partial H}{\partial q_\phi}\end{aligned}}$$

For special relativity:

$$\boxed{\begin{aligned} p_r &= \gamma m \dot{r}, \quad q_r = r, \\ p_\phi &= L, \quad q_\phi = \phi \end{aligned}} \quad -(55)$$

These results are rigorously consistent with the Lagrangian analysis of previous papers, but the Hamiltonian analysis gives additional and elegant results such as

$$\dot{\phi} = \frac{\partial H}{\partial L}; \quad \dot{r} = \frac{\partial H}{\partial (\gamma m \dot{r})} \quad -(56)$$

and

$$\dot{p}_r = -\frac{\partial H}{\partial r} = -\frac{mMG}{r^2} \quad -(57)$$

$$\dot{p}_\phi = \dot{L} = 0 \quad -(58)$$