425(4): Basic Theory and Orbital Hamilton Equations

Acii Maria and Themba chapter 6, the Hamiltonian is defined by the Euler Lagrange equations using:

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}}, \quad \dot{p} = \frac{\partial \mathcal{L}}{\partial q} \quad -(1)$$

Using:

$$\frac{\partial \mathcal{L}}{\partial t} = 0 \quad -(2)$$

then

$$\frac{dL}{dt} = \frac{\partial \mathcal{L}}{\partial q}\dot{q} + \frac{\partial \mathcal{L}}{\partial \dot{q}}\ddot{q} \quad -(3)$$

$$= \dot{q}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) + \frac{\partial \mathcal{L}}{\partial \dot{q}}\ddot{q}$$

Using:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{\partial \mathcal{L}}{\partial q} \quad -(4)$$

it follows that:

$$\frac{d}{dt}\left(\mathcal{L} - \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = 0 \quad -(5)$$

The Hamiltonian is defined as:

$$H = \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L} \quad -(6)$$

so

$$\frac{dH}{dt} = 0 \quad -(7)$$

Therefore:

$$H(q, p, t) = p\dot{q} - \mathcal{L}(q, \dot{q}, t) \quad -(8)$$

The Hamiltonian is a function of q and p, which are canonically conjugate variables:

$$\frac{dp}{dq} = 0 \quad -(9)$$

2) and the Lagrangian is a function of q and $\dot{q}$, in which $\dot{q}$ is the time derivative of q, so q and $\dot{q}$ are not independent:

$$\frac{d\dot{q}}{dq} \neq 0. \quad -(8)$$

The Hamilton equations are derived by considering:

$$dH = \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial p} dp + \frac{\partial H}{\partial t} dt \quad -(9)$$

and

$$H = p\dot{q} - \mathcal{L} \quad -(10)$$

so

$$dH = \dot{q}\, dp + p\, d\dot{q} - \frac{\partial \mathcal{L}}{\partial q} dq - \frac{\partial \mathcal{L}}{\partial \dot{q}} d\dot{q}. \quad -(11)$$

From eqs. (9) and (11):

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} \quad -(12)$$

which are the Hamilton equations, Q.E.D.

These are powerful first order differential equations used in many fields of physics. They are written in terms of generalized coordinates p and q.

The canonically conjugate variables p and q must be defined before the Hamilton equations can be applied. This can be done using the Lagrange equations or by inspection.

For the classical gravitational problem:

$$\mathcal{L} = \frac{p^2}{2m} + \frac{mMG}{r} \quad -(13)$$

and in plane polar coordinates (r, ϕ):

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{mMG}{r} \quad -(14)$$

with:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\partial \mathcal{L}}{\partial r} \quad -(15)$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\partial \mathcal{L}}{\partial \phi} \quad -(16)$$

The generalized momenta are:

$$p_r = p = \frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(17)$$

$$p_\phi = L = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad -(18)$$

and the generalized q are:

$$q_r = r, \quad q_\phi = \phi \quad -(19)$$

The Hamiltonian is:

$$H = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{mMG}{r} \quad -(20)$$

From eq. (17):

$$p_r = m\dot{r}, \quad -(21)$$

and from eq. (18):

$$p_\phi = L = mr^2\dot{\phi} \quad -(22)$$

So

$$H = \frac{1}{2}m(p_r\dot{r} + p_\phi\dot{\phi}) - \frac{mMG}{q} \quad -(23)$$

$$= \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} - \frac{mMG}{q}$$

The first Hamilton equation:

$$\dot{q} = \frac{\partial H}{\partial p} \quad -(24)$$

4) gives
$$\dot{q}_r = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \quad -(25)$$

i.e
$$\boxed{p_r = m\dot{r}} \quad -(26)$$

The first Hamilton equation also gives:
$$\dot{\phi} = \dot{q}_\phi = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2} \quad -(27)$$

so:
$$\boxed{p_\phi = mr^2\dot{\phi}} \quad -(28)$$

Eqs. (26) and (28) mean:
$$\boxed{p = m\dot{r}, \; L = mr^2\dot{\phi}} \quad -(29)$$

The second Hamilton equation:
$$\dot{p} = -\frac{\partial H}{\partial q} \quad -(30)$$

gives:
$$\boxed{\begin{aligned} \dot{p}_r &= -\frac{mMG}{r^2} \quad -(31) \\ \text{and} \quad \dot{p}_\phi &= \dot{L} = 0 \quad -(32) \end{aligned}}$$

Eq. (31) means: $m\ddot{r} = -\frac{mMG}{r^2} \quad -(33)$

which is the orbital force equation in inertial coordinates.

5) The vector formulation of eq. (33) is:

$$m\underline{\ddot{r}} = -\frac{mMG}{r^2}\,\underline{e}_r \quad -(34)$$

where $\underline{e}_r$ is the radial unit vector. In plane polar coordinates:

$$\underline{\ddot{r}} = (\ddot{r} - r\dot{\phi}^2)\,\underline{e}_r + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\,\underline{e}_\phi \quad -(35)$$

so eq. (34) splits into:

$$m(\ddot{r} - r\dot{\phi}^2) = -\frac{mMG}{r^2} \quad -(36)$$

and

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0 \quad -(37)$$

Eq. (37) is equivalent to:

$$\frac{dL}{dt} = 0 \quad -(38)$$

and eq. (36) can be derived from:

$$\frac{dH}{dt} = 0 \quad -(39)$$

Eqs. (38) and (39) are the Evans Eckardt equations.

Note that

$$\dot{p} = \frac{\partial \mathcal{L}}{\partial r} = -\frac{\partial H}{\partial r} \quad -(40)$$

so

$$\frac{\partial(p\dot{r})}{\partial r} = 0 \quad -(41)$$

which implies that

$$\frac{\partial \dot{r}}{\partial r} = 0. \quad -(42)$$

This analysis must now be repeated for the relativistic and m theory