324(2): Geodesic Method of Defining the Hamiltonian

This was first described in UFT416 and is based on the lagrangian:

$$\mathcal{L} = \frac{1}{2} m c^2 = \frac{1}{2} m \left(\frac{ds}{d\tau} \right)^2 = \frac{1}{2} m g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \quad -(1)$$

In the (r, ϕ) coordinate system:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = m(r) c^2 dt^2 - \frac{dr^2}{m(r)} - r^2 d\phi^2 \quad -(2)$$

The Hamilton Principle of Least Action is:

$$\delta \int \mathcal{L} \, d\tau = 0 \quad -(3)$$

where τ is the proper time, and the Euler Lagrange equation is:

$$\frac{d}{d\tau} \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \frac{\partial \mathcal{L}}{\partial x^\mu} \quad -(4)$$

It follows that the total relativistic energy is defined by:

$$E = \frac{\partial \mathcal{L}}{\partial (dt/d\tau)} = m(r) m c^2 \frac{dt}{d\tau}$$

$$= m(r) \gamma m c^2 \quad -(5)$$

where

$$\gamma = \left(m(r) - \frac{\dot{r}^2 + r^2 \dot{\phi}^2}{m(r) c^2} \right)^{-1/2} \quad -(6)$$

The Hamiltonian is therefore:

$$H = E + U \quad -(7)$$

$$= m(r) \gamma m c^2 + U$$

and

$$\frac{dH}{dt} = 0 \quad -(8)$$

This is the first Evans / Eckardt equation.

Defining:

$$r_1 = \frac{r}{m(r)^{1/2}} \quad -(9)$$

the Euler Lagrange equation:

$$\frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial r_1} = \frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial\left(\frac{dr_1}{d\tau}\right)} = 0 \quad -(10)$$

gives the linear momentum:

$$p_1 = m\frac{dr_1}{d\tau} \quad -(11)$$

and the conserved relativistic linear momentum:

$$p = \gamma m \frac{dr}{dt} \quad -(12)$$

Finally the angular momentum is defined by: $-(13)$

$$L = \frac{\partial \mathcal{L}}{\partial\left(\frac{d\phi}{d\tau}\right)} = m r_1^2 \frac{d\phi}{d\tau} = \frac{\gamma m r^2}{m(r)}\frac{d\phi}{dt}$$

<u>The second Evans Eckardt equation is</u>:

$$\frac{dL}{dt} = 0. \quad -(14)$$

The complete momentum in the (r, ϕ) system is:

$$\underline{p}_N = m\underline{\dot{r}} = \dot{r}\,\underline{e}_r + r\dot{\phi}\,\underline{e}_\phi \quad -(15)$$

so the complete relativistic momentum is:

$$\underline{p} = m\gamma\left(\dot{r}\,\underline{e}_r + r\dot{\phi}\,\underline{e}_\phi\right) = \gamma\underline{p}_N \quad -(16)$$

where $\underline{p}_N$ is the Newtonian momentum.

In m space the non-relativistic momentum is:

$$\underline{p}_0 = \frac{m}{m(r)^{1/2}}\left(\dot{r}\,\underline{e}_r + r\dot{\phi}\,\underline{e}_\phi\right) \quad -(17)$$

and the relativistic momentum is:

$$\underline{p} = \frac{\gamma m}{n(r)^{1/2}} \left(\dot{r}\, \underline{e}_r + r \dot{\phi}\, \underline{e}_\phi \right) \quad - (18)$$

It follows that:

$$p_0^2 = \frac{m^2}{n(r)} \left(\dot{r}^2 + r^2 \dot{\phi}^2 \right)$$

$$= \frac{m^2}{n(r)} \left(\dot{r}^2 + \frac{L^2}{r^2} \right) \quad - (19)$$

It follows that dynamics and cosmology can be evaluated with

$$\frac{dH}{dt} = 0 \quad - (20)$$

and

$$\frac{dL}{dt} = 0 \quad - (21)$$

and that this is the most fundamental method.