420(6): Combined Use of the Hamiltonian and Lagrangian Methods

The Hamiltonian method gives:

$$\frac{d}{dt}\left(\gamma m v_1\right) = -\frac{mMG}{r_1^2} - mc^2\gamma\frac{dn(r_1)}{dr_1} \quad -(1)$$

and the Lagrangian method gives:

$$\frac{d}{dt}\left(\gamma m v_1\right) = -\frac{mMG}{r_1^2} - \frac{mc^2}{2}\gamma\frac{dn(r_1)}{dr_1} \quad -(2)$$

where

$$\gamma = \left(n(r_1) - \frac{v_1^2}{c^2}\right)^{-1/2} \quad -(3)$$

It is considered that the Hamiltonian method:

$$\frac{dH}{dt} = 0 \quad -(4)$$

gives the correct result. It is also the simpler of the two methods. The Lagrangian used in deriving Eq. (2) in Note 417(1) is:

$$\mathcal{L} = -mc^2\left(n(r_1) - \frac{1}{c^2}\dot{\underline{r}}_1\cdot\dot{\underline{r}}_1\right)^{1/2} + \frac{mMG}{r_1} \quad -(5)$$

where

$$r_1 = \frac{r}{n(r)^{1/2}} \quad -(6)$$

In Cartesian coordinates:

$$\underline{v}_1 = \dot{\underline{r}}_1. \quad -(7)$$

The Euler Lagrange equation is:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\underline{r}}_1} = \frac{\partial\mathcal{L}}{\partial\underline{r}_1} = \underline{\nabla}\mathcal{L} = \frac{\partial\mathcal{L}}{\partial r_1}\underline{e}_r \quad -(8)$$

which gives

$$\underline{F}_1 = \frac{d}{dt}\left(\gamma m \underline{\dot{r}}_1\right) = \frac{\partial \mathcal{L}}{\partial r_1}\underline{e}_r \quad -(9)$$

in which:

$$\underline{\dot{r}}_1 = \dot{r}_1 \underline{e}_r + r_1\dot{\phi}\,\underline{e}_\phi \quad -(10)$$

so

$$\frac{d}{dt}\left(\gamma m \dot{r}_1\right) = \frac{\partial \mathcal{L}}{\partial r_1} \quad -(11)$$

and

$$\frac{d}{dt}\left(\gamma m r_1 \dot{\phi}\right) = 0 \quad -(12)$$

Eq. (12) is

$$\frac{dL}{dt} = 0 \quad -(13)$$

where

$$L = \gamma m r_1^2 \dot{\phi} \quad -(14)$$

is the conserved angular momentum. So eq. (13) is given correctly by the choice of lagrangian (5).

However:

$$\frac{\partial \mathcal{L}}{\partial r_1} = -\frac{mMG}{r_1^2} - \frac{mc^2}{2}\gamma\frac{dm(r_1)}{dr_1} \quad -(15)$$

and eq. (2) is defined in the inertial system:

$$v_1 = \dot{r}_1 \quad -(16)$$

Eqs. (1) and (2) are both written in an inertial Cartesian system.

Therefore the lagrangian must be changed to:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 \quad -(17)$$

where

$$\frac{\partial \mathcal{L}_1}{\partial r_1} = -\frac{mc^2}{2}\gamma\frac{dm(r_1)}{dr_1} \quad -(18)$$

and
$$\frac{\partial \mathcal{L}_1}{\partial \dot{r}_1} = 0 \quad -(19)$$

A lagrangian of the type (17) to (19) gives eq. (1) using:
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}_1} = \frac{\partial \mathcal{L}}{\partial r_1} \quad -(20)$$
where
$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 \quad -(21)$$

This is a formal solution of the problem, provided eqs. (18) and (19) can be integrated.

A possible solution of eq. (18) is:
$$\mathcal{L}_1 = -\frac{mc^2}{2}\gamma\, m(r_1) \quad -(22)$$
so eq. (19) implies:
$$\frac{d}{\partial \dot{r}_1}\left(\gamma\, m(r_1)\right) = 0 \quad -(23)$$
In Cartesian coordinates:
$$\gamma = \left(m(r_1) - \frac{\dot{r}_1^{\,2}}{c^2} \right)^{-1/2} \quad -(24)$$
So
$$\frac{\partial \gamma}{\partial \dot{r}_1} = \frac{\dot{r}_1}{c^2}\gamma^3 \quad -(25)$$
Therefore eq. (23) is:
$$m(r_1)\frac{\partial \gamma}{\partial \dot{r}_1} + \gamma\frac{\partial m(r_1)}{\partial \dot{r}_1} = 0 \quad -(26)$$
i.e.

$$\frac{\dot{r}_1}{c^2}\gamma^3 + \gamma\frac{\partial m(r_1)}{\partial \dot{r}_1} = 0 \quad -(27)$$

or:

$$\boxed{\frac{\partial m(r_1)}{\partial \dot{r}_1} = -\frac{\dot{r}_1}{c^2}\gamma^2} \quad -(28)$$

Therefore the new constraint (28) emerges.
The lagrangian is therefore:

$$\mathcal{L} = -mc^2\left(m(r_1) - \frac{\dot{r}_1^2}{c^2}\right)^{1/2} + \frac{mMG}{r_1} - \frac{mc^2}{2}\gamma m(r_1)$$

$$= -mc^2\left(\frac{1}{\gamma} + \frac{\gamma}{2}m(r_1)\right) + \frac{mMG}{r_1} \quad -(29)$$

constrained by:

$$\frac{\partial m(r_1)}{\partial \dot{r}_1} = -\frac{\dot{r}_1}{c^2}\gamma^2 \quad -(30)$$

i.e.

$$\frac{\partial m(r_1)}{\partial v_1} = -\frac{v_1}{c^2}\gamma^2 \quad -(31)$$

As in Eq. (14) of Note 420(5):

$$\frac{dm(r_1)}{dr_1} = \frac{dm(r_1)}{dt}\frac{dt}{dr_1} = \frac{1}{v_1}\frac{dm(r_1)}{dt} \quad -(32)$$

and

$$\frac{dm(r_1)}{dt} = \frac{dm(r)}{dv_1}\dot{v}_1 \quad -(33)$$

so:

$$\frac{dm(r_1)}{dv_1} = \frac{v_1}{v_1}\frac{dm(r)}{dv_1} \quad -(34)$$

From eqs. (31) and (34):

$$\boxed{\frac{dm(r_1)}{dv_1} = -\frac{\dot{v}_1}{c^2}\gamma^2} \quad -(35)$$

Eq. (35) is a new equation for $dm(r_1)/dv_1$, showing that it is a small quantity. A constant $m(r_1)$ theory is therefore a very good approximation.