420(7): The Evans Eckardt Equations of Motion of Classical Dynamics

These equations are very simple and fundamental and are defined by the fact that the hamiltonian H and angular momentum L are constants of motion. It has been shown in previous notes that they give the same result as the lagrangian method in classical dynamics and special relativity. They are:

$$\frac{dH}{dt} = 0 \quad -(1)$$

and

$$\frac{dL}{dt} = 0 \quad -(2)$$

They begin to show their power in m theory, where:

$$H = m(r_1)\, mc^2 \gamma - \frac{mMG}{r_1} \quad -(3)$$

and

$$L = \frac{\gamma m r^2 \dot{\phi}}{m(r_1)} \quad -(4)$$

in the coordinate system (r_1, ϕ), where:

$$r_1 = \frac{r}{m(r)^{1/2}} \quad -(5)$$

The major discovery has already been made that in m theory, the Lorentz factor is generalized to:

$$\gamma = \left(m(r_1) - \frac{\dot{\underline{r}}_1 \cdot \dot{\underline{r}}_1}{c^2} \right)^{-1/2} \quad -(6)$$

In the system (r_1, ϕ):

$$\dot{\underline{r}}_1 \cdot \dot{\underline{r}}_1 = \dot{r}_1^2 + r_1^2 \phi^2 \quad -(7)$$

So the equation of motion (1) is:

$$-\frac{c^2}{2} m m(r_1)\gamma^{3/2}\left(\frac{dm(r_1)}{dt}-\frac{1}{c^2}\left(2\frac{dr_1}{dt}\right)\left(\frac{d^2 r_1}{dt^2}\right.\right.$$

$$\left.\left.+2\left(\frac{d\phi}{dt}\right)^2\left(r_1\frac{dr_1}{dt}\right)+2\left(\frac{d\phi}{dt}\right)\left(\frac{d^2\phi}{dt^2}\right)r_1^2\right)\right)$$

$$+mc^2\frac{dm(r_1)}{dt}\gamma-\frac{mMG}{r_1 m(r_1)^{1/2}}\frac{dm(r_1)}{dt} \quad -(8)$$

$$+\frac{mMG}{r_1^2}m(r_1)^{1/2}\frac{dr_1}{dt}=0$$

Now use:

$$\frac{dm(r_1)}{dt}=\frac{dr_1}{dt}\frac{dm(r_1)}{dr_1} \quad -(9)$$

and computer algebra to find the result:

$$\frac{d^2 r_1}{dt^2}=\frac{c^2}{2}\frac{dm(r_1)}{dr_1}\left(1-\frac{1}{\gamma^2 m(r_1)}\right)$$

$$-\left(\frac{d\phi}{dt}\right)^2 r_1-\frac{\left(\frac{d\phi}{dt}\right)\left(\frac{d^2\phi}{dt^2}\right)r_1^2}{\frac{dr_1}{dt}} \quad -(10)$$

$$-\frac{MG}{\gamma^3 r_1^2(m(r_1))^{1/2}}\left(1-\frac{dm(r_1)}{dr_1}\frac{r_1}{m(r_1)}\right)$$

Similarly:

$$\frac{dL}{dt}=0, \quad -(11)$$

the conservation of angular momentum, gives:

$$\frac{d^2\phi}{dt^2} = \frac{\gamma^2 c^2 \frac{d\phi}{dt} r_1 \frac{dr_1}{dt} \frac{dm(r_1)}{dr_1}}{2\gamma^2\left(\frac{d\phi}{dt}\right)^2 r_1^3 + 2c^2 r_1} - \frac{2\gamma^2\left(\frac{d\phi}{dt}\right) r_1 \frac{dr_1}{dt}\frac{d^2 r_1}{dt^2}}{2\gamma^2\left(\frac{d\phi}{dt}\right)^2 r_1^3 + 2c^2 r_1}$$

$$- \frac{2\gamma^2\left(\frac{d\phi}{dt}\right)^3 r_1^2\left(\frac{dr_1}{dt}\right)}{2\gamma^2\left(\frac{d\phi}{dt}\right)^2 r_1^3 + 2c^2 r_1} + \frac{4c^2\left(\frac{d\phi}{dt}\right)\frac{dr_1}{dt}}{2\gamma^2\left(\frac{d\phi}{dt}\right)^2 r_1^3 + 2c_1^2 r_1} \quad -(12)$$

Eqs. (10) and (12) are integrated simultaneously to give classical dynamics in n space.

Limit of Newtonian Dynamics

In Cartesian coordinates:

$$H = \frac{1}{2} m \dot{r}^2 - \frac{mMG}{r} \quad -(13)$$

so

$$\frac{dH}{dt} = 0 \quad -(14)$$

gives:

$$\frac{1}{2} m \frac{d}{dt}\dot{r}^2 = -mMG\frac{d}{dt}\left(\frac{1}{r}\right) \quad -(14)$$

Use:

$$\frac{d}{dt}\dot{r}^2 = \frac{d}{d\dot{r}}\dot{r}^2\,\ddot{r} = 2\dot{r}\ddot{r} \quad -(15)$$

$$\frac{d}{dt}\left(\frac{1}{r}\right) = \frac{d}{dr}\left(\frac{1}{r}\right)\dot{r} = -\frac{\dot{r}}{r^2} \quad -(16)$$

to find

$$\ddot{r} = -\frac{MG}{r^2} \quad -(17)$$

4) which is the Newtonian result in Cartesian coordinates, Q.E.D.

In plane polar coordinates:

$$H = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2 \right) - \frac{mMG}{r} \quad -(18)$$

so

$$\dot{r}\ddot{r} + \frac{1}{2} \frac{d}{dt} \left(r^2 \dot{\phi}^2 \right) = - \frac{MG}{r^2} \dot{r} \quad -(19)$$

$$-(20)$$

Using the Leibniz Theorem:

$$\dot{r}\ddot{r} + \frac{1}{2} \left(\dot{\phi}^2 \frac{d}{dt} r^2 + r^2 \frac{d}{dt} \dot{\phi}^2 \right) = - \frac{MG\dot{r}}{r^2}$$

Now use:

$$\frac{d}{dt} \dot{\phi}^2 = \frac{d}{d\dot{\phi}} \dot{\phi}^2 \frac{d\dot{\phi}}{dt} = 2 \dot{\phi} \ddot{\phi} \quad -(21)$$

$$-(22)$$

to find that

$$\dot{r}\ddot{r} + r\dot{\phi} \left(\dot{r}\dot{\phi} + r\ddot{\phi} \right) = - \frac{MG}{r^2} \dot{r}$$

Finally use

$$\frac{dL}{dt} = 0 \quad -(23)$$

to find that

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0 \quad -(24)$$

From eqs. (22) and (24):

$$\ddot{r} - r\dot{\phi}^2 = - \frac{MG}{r^2} \quad -(25)$$

which is the force equation in plane polar coordinates, Q.E.D.

Eq. (25) is also given by the Lagrangian:

$$\mathcal{L} = m \left(\dot{r}^2 + r^2 \dot{\phi}^2 \right) + \frac{mMG}{r} \quad -(26)$$

) and the Euler Lagrange equations:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\partial \mathcal{L}}{\partial r} \quad -(27)$$

and

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\partial \mathcal{L}}{\partial \phi} \quad -(28)$$

Eq. (27) gives eq. (25) and eq. (28) gives:

$$\frac{dL}{dt} = 0 \quad -(29)$$

where

$$L = mr^2\dot{\phi} \quad -(30)$$

Eqs. (29) and (30) give eq. (24). On the non-relativistic classical level the lagrangian and hamiltonian methods give the same result, as is well known.

Note that it is much easier to use:

$$H = \frac{1}{2}mv^2 - \frac{mMG}{r} \quad -(31)$$

and

$$\mathcal{L} = \frac{1}{2}mv^2 + \frac{mMG}{r} \quad -(32)$$

From eq. (1):

$$\frac{1}{2}m\frac{dv^2}{dt} = -\frac{mMG}{r^2}\dot{r} \quad -(33)$$

so

$$\dot{v}v = -\frac{mMG}{r^2}\dot{r} \quad -(34)$$

and use

$$\dot{v} = \ddot{r} - r^2\dot{\phi} \quad -(35)$$

$$v = \dot{r} \quad -(36)$$

to get eq. (25).

The Euler Lagrange equation (37) becomes:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v} = \frac{\partial \mathcal{L}}{\partial r} \quad -(37)$$

and the Lagrangian becomes:

$$\mathcal{L} = \frac{1}{2} m v^2 + \frac{m M G}{r} \quad -(38)$$

Eqs (37) and (38) give:

$$\dot{v} = \ddot{r} - r^2 \dot{\phi} = -\frac{MG}{r^2} \quad -(39)$$

Q.E.D.

Therefore as in Note 420(5) it is easier to define the Hamiltonian as:

$$H = n(r_1)\gamma m c^2 - \frac{m M G}{r_1} \quad -(40)$$

here

$$\gamma = \left(n(r_1) - \frac{v_1^2}{c^2} \right)^{-1/2} \quad -(41)$$

and

$$v_1^2 = \dot{r}_1^2 + r_1^2 \dot{\phi}^2 \quad -(42)$$

From eqs. (1) and (40):

$$m c^2 \frac{d}{dt}\left(n(r_1)\gamma \right) = \frac{d}{dt}\frac{m M G}{r_1} = -\frac{m M G}{r_1^2}\dot{r}_1 \quad -(43)$$

Now use:

$$\frac{d}{dt}\left(n(r_1)\gamma \right) = \frac{d}{dv_1}\left(n(r_1)\gamma \right)\dot{v}_1 \quad -(44)$$

The equation of motion is:

$$mc^2 \frac{d}{dV_1}\left(n(r_1)\gamma\right)\dot{V}_1 = -\frac{mMG}{r_1^2}\,\dot{r}_1 \quad -(45)$$

Using the Leibniz Theorem:

$$\frac{d}{dV_1}\left(n(r_1)\gamma\right) = \gamma\frac{dn(r_1)}{dV_1} + n(r_1)\frac{d\gamma}{dV_1} \quad -(46)$$

Here

$$\frac{d\gamma}{dV_1} = \frac{v_1}{c^2}\gamma^3 \quad -(47)$$

$$-(47)$$

So

$$\left(n(r_1)v_1\gamma^3 + c^2\gamma\frac{dn(r_1)}{dV_1}\right)\frac{dv_1}{dt} = -\frac{mMG}{r_1^2}\,r_1$$

Now use:

$$\frac{d}{dt}\left(\gamma v_1\right) = n(r)\gamma^3\frac{dV_1}{dt} \quad -(48)$$

in the equation of motion:

$$n(r_1)v_1\gamma^3\frac{dv_1}{dt} + c^2\gamma\frac{dn(r_1)}{dV_1}\frac{dV_1}{dt} = -\frac{mMG}{r_1^2}\,\dot{r}_1$$

$$-(49)$$

Now use:

$$v_1 = \dot{r}_1 \quad -(50)$$

so

$$\frac{d}{dt}\left(\gamma m\dot{r}_1\right) = -\frac{mMG}{r_1^2} - \frac{mc^2}{V_1}\frac{dV_1}{dt}\frac{dn(r_1)}{dV_1}\gamma$$

$$-(51)$$

Finally use:

8)

$$\frac{dn(r_1)}{dv_1}\frac{dv_1}{dt} = \frac{dn(r_1)}{dt} \quad -(52)$$

$$-(53)$$

and

$$\frac{1}{v_1}\frac{dn(r_1)}{dt} = \frac{dn(r_1)}{dr_1}\frac{dr_1}{dt}\frac{1}{v_1} = \frac{dn(r_1)}{dr_1}$$

to find that:

$$\boxed{\frac{d}{dt}\left(\gamma m \dot{r}_1\right) = -\frac{nmG}{r_1^2} - nc^2\gamma\frac{dn(r_1)}{dr_1}} \quad -(54)$$

Eqs. (10) and (54) are equivalent provided that

$$v_1^2 = \dot{r}_1^2 + r_1^2\dot{\phi}^2 \quad -(55)$$

However, eq. (54) is derived in the Cartesian system, and eq. (10) in the plane polar system.

The conservation of angular momentum is

$$\frac{dL}{dt} = \frac{d}{dt}\left(\frac{\gamma m r_1^2\dot{\phi}}{n(r_1)}\right) = 0 \quad -(56)$$

in plane polar coordinates.

In Cartesian coordinates the angular momentum is

$$\underline{L} = \underline{r}\times\underline{p} = m\underline{r}\times\underline{v} \quad -(57)$$

on the classical level and

$$\underline{L} = \gamma m\,\underline{r}\times\underline{v} \quad -(58)$$

in m theory

$$= m(r_1)\gamma\,\underline{r_1}\times\underline{v_1}$$

In Cartesian coordinates:

$$\underline{v_1} = \underline{\dot{r}_1} \quad -(59)$$

So
$$\underline{L} = m(r_1)\gamma\, \underline{r_1} \times \underline{\dot{r}_1} \quad -(60)$$
If motion takes place in the XY plane, as in an orbit, then:
$$\underline{L} = m(r_1)\gamma\left(x\dot{y} - y\dot{x}\right)\underline{k} \quad -(61)$$
and
$$L = m(r_1)\gamma\left(x\dot{y} - y\dot{x}\right) \quad -(62)$$
Finally define:
$$L_1 = \frac{L}{m(r_1)} \quad -(63)$$
so
$$L_1 = \gamma\left(x\dot{y} - y\dot{x}\right) \quad -(64)$$
and
$$\frac{dL_1}{dt} = 0 \quad -(65)$$
The complete equations (10) and (12) give the most information.
