15(5): Double Cross Check with the Lagrangian Method

Note that the correct Lagrange variable is $\underline{r}$, the Euler Lagrange equation is: — (1)

$$\frac{\partial \mathcal{L}}{\partial \underline{r}} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\underline{r}}}$$

— (2)

with the Lagrangian:

$$\mathcal{L} = -mc^2\left(m(r) - \frac{\dot{\underline{r}}\cdot\dot{\underline{r}}}{c^2}\right)^{-1/2} + \frac{mMG}{r^2}\underline{r}$$

It follows that: — (3)

$$\frac{d}{dt}\left(\gamma m \dot{\underline{r}}\right) = -\frac{mMG}{r^3}\underline{r}$$

$$= -\frac{mMG}{r^2}\underline{e}_r$$

This is the same as eq. (27) of Note 415(4), Q.E.D., giving a double cross check.

Eq. (3) gives:

$$\frac{d\gamma}{dt}\dot{r} + \gamma\left(\ddot{r} - r\dot{\phi}^2\right) = -m(r)^{1/2}\frac{MG}{r^2} \quad -(4)$$

and

$$\frac{d\gamma}{dt}r\dot{\phi} + r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0 \quad -(5)$$

Adding the spin connection term to Eq. (4) gives:

$$\frac{d\gamma}{dt}\dot{r} + \gamma\left(\ddot{r} - r\dot{\phi}^2\right) = -m(r)^{1/2}\frac{MG}{r}\left(\frac{1}{r} + \Omega_r\right) \quad -(6)$$

Solving eqs. (5) and (6) simultaneously gives the orbit.

The conserved angular momentum is:

$$\underline{L} = \frac{\gamma m r^2}{n(r)} \dot{\phi} \underline{k} \quad -(7)$$

So

$$\boxed{L = \frac{\gamma m r^2}{n(r)} \dot{\phi}} \quad -(8)$$

Eq. (6) can be simplified using:

$$\dot{r} = v, \quad \frac{dv}{dt} = \ddot{r} - r\dot{\phi}^2 \quad -(9)$$

so

$$F = m\left(v\frac{d\gamma}{dt} + \gamma\frac{dv}{dt}\right) = \frac{d}{dt}(\gamma m v) \quad -(10)$$

Using

$$\gamma = \left(n(r) - \frac{v^2}{c^2}\right)^{-1/2} \quad -(11)$$

it follows that:

$$\frac{d\gamma}{dv} = \gamma^3 \frac{v}{c^2} \quad -(12)$$

so

$$F = m\gamma\frac{dv}{dt}\left(1 + \gamma^2\frac{v^2}{c^2}\right) \quad -(13)$$

$$= m\gamma\frac{dv}{dt}\left(1 + \frac{v^2}{c^2\left(n(r) - \frac{v^2}{c^2}\right)}\right)$$

$$= m\gamma\frac{dv}{dt}\left(\frac{c^2 n(r)}{c^2\left(n(r) - \frac{v^2}{c^2}\right)}\right)$$

$$= \gamma^3 m\, n(r) \frac{dv}{dt}$$

So Eq. (6) simplifies to:

$$\gamma^3 m \frac{dv}{dt} = -\frac{1}{n(r)^{1/2}} \frac{mMG}{r}\left(\frac{1}{r} + \Omega_r\right) \quad -(14)$$

This is the relativistic Newton equation in n space.

Solving eqs. (5) and (14) gives the orbit in n space in terms of n(r) and the spin connection Ω_r.

In eq. (14):

$$\gamma = \left(n(r) - \frac{v^2}{c^2}\right)^{-1/2} \quad -(15)$$

In the limit:

$$\left.\begin{array}{l} n(r) \rightarrow 1, \\ \Omega(r) \rightarrow 0, \end{array}\right\} \quad -(16)$$

$v \ll c$,

Eq. (14) reduces to the usual Newton equation:

$$m \frac{dv}{dt} = -\frac{mMG}{r^2} \quad -(17)$$

Q.E.D.