415(4): The Position Vector in m Space and Double Check

Consider the equation:

$$d\underline{r} \cdot d\underline{r} = \frac{dr^2}{m(r)} + r^2 d\phi^2 \quad -(1)$$

in the infinitesimal line element of m space:

$$ds^2 = c^2 d\tau^2 = m c^2 dt^2 - d\underline{r} \cdot d\underline{r} \quad -(2)$$

In general, m is a function of $\underline{r}$ and t, but we simplify the theory by considering:

$$m := m(\underline{r}) \quad -(3)$$

In eq. (1):

$$d\underline{r} = \frac{\partial \underline{r}}{\partial r} dr + \frac{\partial \underline{r}}{\partial \phi} d\phi \quad -(4)$$

"Vector Analysis Problem Solver").

Therefore:

$$d\underline{r} \cdot d\underline{r} = \left(\frac{\partial \underline{r}}{\partial r} dr + \frac{\partial \underline{r}}{\partial \phi} d\phi \right) \cdot \left(\frac{\partial \underline{r}}{\partial r} dr + \frac{\partial \underline{r}}{\partial \phi} d\phi \right)$$

$$= \frac{dr^2}{m(r)} + r^2 d\phi^2 \quad -(5)$$

One possible solution is:

$$\left(\frac{\partial \underline{r}}{\partial r} \right)^2 dr^2 = \frac{1}{m(r)} dr^2 \quad -(6)$$

$$\left(\frac{\partial \underline{r}}{\partial \phi} \right)^2 d\phi^2 = r^2 d\phi^2 \quad -(7)$$

and

$$\frac{\partial \underline{r}}{\partial \phi} \cdot \frac{\partial \underline{r}}{\partial r} = 0 \quad -(8)$$

so

$$\frac{\partial \underline{r}}{\partial r} = \frac{1}{m(r)} \underline{e}_r \quad -(9)$$

$$\frac{\partial \underline{r}}{\partial \phi} = r\,\underline{e}_\phi \quad -(10)$$

and

$$\frac{\partial \underline{r}}{\partial \phi} \cdot \frac{\partial \underline{r}}{\partial r} = \frac{r}{n(r)^{1/2}}\,\underline{e}_r \cdot \underline{e}_\phi = 0 \quad -(11)$$

So

$$\boxed{\underline{r} = \frac{r}{n(r)^{1/2}}\,\underline{e}_r} \quad -(12)$$

Q.E.D.

It follows that:

$$\dot{\underline{r}} = \frac{d}{dt}\left(\frac{r}{n(r)^{1/2}}\right)\underline{e}_r + \frac{r}{n(r)^{1/2}}\,\dot{\underline{e}}_r \quad -(13)$$

in which

$$\dot{\underline{e}}_r = \dot{\phi}\,\underline{e}_\phi \quad -(14)$$

so

$$\boxed{\underline{v} = \dot{\underline{r}} = \frac{d}{dt}\left(\frac{r}{n(r)^{1/2}}\right)\underline{e}_r + \frac{r\dot{\phi}}{n(r)^{1/2}}\,\underline{e}_\phi} \quad -(15)$$

in which:

$$\frac{d}{dt}\left(\frac{r}{n(r)^{1/2}}\right) = \frac{1}{n(r)^{1/2}}\,\dot{r} + r\,\frac{d}{dt}\left(\frac{1}{n(r)^{1/2}}\right) \quad -(16)$$

$$= \frac{1}{n(r)^{1/2}}\,\dot{r}$$

because in eq. (3) it has been assumed that $n(\underline{r})$ is not a function of t.

It follows that:

$$\underline{\dot{r}} = \frac{1}{n(r)^{1/2}} \left(\dot{r}\,\underline{e}_r + r\dot{\phi}\,\underline{e}_\phi \right) \quad -(17)$$

The relativistic momentum is therefore:

$$\underline{p} = \gamma m \underline{\dot{r}} \quad -(18)$$

where

$$\gamma = \left(n(r) - \frac{\underline{\dot{r}} \cdot \underline{\dot{r}}}{c^2} \right)^{-1/2} \quad -(19)$$

the relativistic angular momentum is therefore:

$$L = \underline{r} \times \underline{p} \quad -(20)$$

$$= \gamma m \begin{vmatrix} \underline{e}_r & \underline{e}_\phi \\ \frac{r}{m(r)^{1/2}} & 0 \\ \frac{\dot{r}}{m(r)^{1/2}} & \frac{r\dot{\phi}}{m(r)^{1/2}} \end{vmatrix}$$

$$\boxed{L = \frac{\gamma m r^2}{m(r)} \dot{\phi}} \quad -(21)$$

and:

$$\frac{dL}{dt} = 0 \quad -(22)$$

<u>This is the first equation of motion of the orbit.</u>

From eqs. (21) and (22):

$$\frac{d}{dt}\left(\gamma r^2 \dot{\phi} \right) = 0 \quad -(23)$$

i.e.

$$\frac{d\gamma}{dt} r\dot{\phi} + \gamma\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right) = 0 \quad -(24)$$

which looks the same as the conservation of angular momentum in UFT 414. However, in m space:

$$\gamma = \left(m(r) - \frac{\dot{r}^2 + r^2\dot{\phi}^2}{m(r)c^2} \right)^{-1/2} \quad -(25)$$

and in the space of UFT 414:

$$\gamma = \left(1 - \frac{\dot{r}^2 + r^2\dot{\phi}^2}{c^2} \right)^{-1/2} \quad -(26)$$

The second equation of motion of the orbit is:

$$\underline{g} = \frac{d}{dt}\left(\gamma \underline{\dot{r}}\right) = \left(-\frac{mG}{r^2} + \Omega_r \Phi \right) \underline{e}_r \quad -(27)$$

$$= -\frac{mG}{r}\left(\frac{1}{r} + \Omega_r \right) \underline{e}_r$$

here Ω_r is the spin connection.

From eqs. (17) and (27):

$$\underline{g} = \frac{d\gamma}{dt}\underline{\dot{r}} + \gamma \underline{\ddot{r}}$$

$$= \frac{d\gamma}{dt}\underline{\dot{r}} + \frac{\gamma}{m(r)^{1/2}}\left(\left(\ddot{r} - r\dot{\phi}^2\right)\underline{e}_r + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\underline{e}_\phi \right)$$

$$= \frac{1}{m(r)^{1/2}}\left[\frac{d\gamma}{dt}\left(\dot{r}\,\underline{e}_r + r\dot{\phi}\,\underline{e}_\phi\right) + \gamma\left(\left(\ddot{r} - r\dot{\phi}^2\right)\underline{e}_r + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\underline{e}_\phi\right) \right]$$

From eqs. (27) and (28) it follows that:

$$\frac{1}{n(r)^{1/2}}\left[\frac{d\gamma}{dt}\dot{r}+\gamma\left(\ddot{r}-r\dot{\phi}^2\right)\right] = -\frac{mG}{r}\left(\frac{1}{r}+\mathcal{L}_r\right) \quad -(29)$$

i.e.

$$\frac{d\gamma}{dt}\dot{r}+\gamma\left(\ddot{r}-r\dot{\phi}^2\right) = -n(r)^{1/2}\frac{mG}{r}\left(\frac{1}{r}+\mathcal{L}_r\right) \quad -(30)$$

which is the Leibniz equation in m space.

As in UFT414 the Leibniz equation in the usual space is:

$$\frac{d\gamma}{dt}\dot{r}+\gamma\left(\ddot{r}-r\dot{\phi}^2\right) = -\frac{mG}{r^2} \quad -(31)$$

The γ in eq. (30) is:

$$\gamma = \left(1-\frac{v^2}{c^2}\right)^{-1/2} \quad -(32)$$

and the γ in eq. (31) is

$$\gamma = \left(n(r)-\frac{v^2}{n(r)c^2}\right)^{-1/2} \quad -(33)$$

As in Note 414(4), eq. (31) may be written as:

$$\gamma^3\frac{dv}{dt} = -\frac{mG}{r^2} \quad -(34)$$

where

$$v = \dot{r} \quad -(35)$$

and

$$\frac{dv}{dt} = \ddot{r}-r\dot{\phi}^2 \quad -(36)$$

Eq. (34) is the relativistic Newton equation.

6) From eqs. (27) and (28):

$$\frac{d\gamma}{dt} r\dot{\phi} + r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0 \quad - (37)$$

which is eq. (24) Q.E.D.

This provides a double cross check on the relativistic angular momentum of the theory, eq. (21), Q.E.D.