#15(8): Lagrangian Method.

By definition: $\underline{r} = \dfrac{r}{m(r)^{1/2}} \underline{e}_r$ — (1)

and $\underline{\dot{r}} = \dfrac{1}{m(r)^{1/2}} \left(\dot{r} \underline{e}_r + r\dot{\phi} \underline{e}_\phi \right)$ — (2)

The relativistic momentum is:

$$\underline{p} = \gamma m \underline{\dot{r}} \quad -(3)$$

where

$$\gamma = \left(m(r) - \frac{\underline{\dot{r}} \cdot \underline{\dot{r}}}{c^2} \right)^{-1/2} \quad -(4)$$

By definition:

$$\underline{p} = \frac{\partial \mathcal{L}}{\partial \underline{\dot{r}}} \quad -(5)$$

where $\mathcal{L}$ is the Lagrangian. Choose:

$$\mathcal{L} = -mc^2 \left(m(r) - \frac{\underline{\dot{r}} \cdot \underline{\dot{r}}}{c^2} \right)^{1/2} + \frac{mMG}{r} \quad -(6)$$

and use

$$U = -\frac{mMG}{r} \quad -(7)$$

The Euler Lagrange equations are:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \underline{\dot{r}}} = \frac{\partial \mathcal{L}}{\partial \underline{r}} \quad -(8)$$

and from Eq. (6):

$$\underline{p} = \frac{\partial \mathcal{L}}{\partial \underline{\dot{r}}} = \gamma m \underline{\dot{r}} \quad -(9)$$

Q.E.D.

So

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \underline{\dot{r}}} \right) = m \frac{d\gamma}{dt} \underline{\dot{r}} + m\gamma \underline{\ddot{r}} \quad -(10)$$

Note that:

$$\frac{\partial}{\partial \underline{r}}\left(\frac{1}{r}\right) = \frac{\partial}{\partial \underline{r}}\left(\underline{r}\cdot\underline{r}\right)^{-1/2}$$

$$= 2\underline{r}\left(-\frac{1}{2}\left(\underline{r}\cdot\underline{r}\right)^{-3/2}\right) \quad -(11)$$

$$= \frac{-\underline{r}}{\left(\underline{r}\cdot\underline{r}\right)^{3/2}} = -\frac{\underline{r}}{r^3} = -\frac{1}{r^2}\,\underline{e}_r$$

So:

$$m\frac{d\gamma}{dt}\,\dot{\underline{r}} + m\gamma\,\ddot{\underline{r}} = -\frac{mMG}{r^2}\,\underline{e}_r \quad -(12)$$

Here:

$$\dot{\underline{r}} = \frac{1}{m(r)^{1/2}}\left(\dot{r}\,\underline{e}_r + r\dot{\phi}\,\underline{e}_\phi\right) \quad -(13)$$

and

$$\ddot{\underline{r}} = \frac{1}{n(r)^{1/2}}\frac{d}{dt}\left(\dot{r}\,\underline{e}_r + r\dot{\phi}\,\underline{e}_\phi\right) \quad -(14)$$

Here:

$$\dot{\underline{r}} = \frac{1}{n(r)^{1/2}}\left(\dot{r}\,\underline{e}_r + r\dot{\phi}\,\underline{e}_\phi\right) \quad -(15)$$

and

$$\ddot{\underline{r}} = \frac{1}{n(r)^{1/2}}\left(\left(\ddot{r} - r\dot{\phi}^2\right)\underline{e}_r + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\underline{e}_\phi\right) \quad -(16)$$

It follows that:

$$\frac{1}{n(r)^{1/2}}\left(\frac{d\gamma}{dt}\,\dot{r} + \gamma\left(\ddot{r} - r\dot{\phi}^2\right)\right) \quad -(17)$$

$$= -\frac{MG}{r}\left(\frac{1}{r} + \Omega_r\right)$$

where Ω_r has been added, and

$$\frac{1}{n(r)^{1/2}}\left(\frac{d\gamma}{dt}\,r\dot{\phi} + r\ddot{\phi} + 2\dot{r}\dot{\phi}\right) = 0 \quad -(18)$$

By definition:

$$L = |\mathbf{r} \times \mathbf{p}| = \frac{\gamma m r^2}{m(r)} \dot{\phi} \quad -(11)$$

It follows from eq. (18) that:

$$\frac{dL}{dt} = 0 \quad -(12)$$

The angular momentum is therefore defined as:

$$\mathbf{L} := \frac{1}{m(r)^{1/2}} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}} = \mathbf{r} \times \mathbf{p} \quad -(13)$$

and the force equation is:

$$\mathbf{F} = \frac{d}{dt}\left(\gamma m \dot{\mathbf{r}}\right) = -\frac{mMG}{r^2}\mathbf{e}_r + \mathbf{\Omega}\,\mathbf{\Phi} \quad -(14)$$

In order to clarify the meaning of eq. (8), note that

$$\mathbf{p} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}} = \frac{\gamma m}{m(r)^{1/2}}\left(\dot{r}\,\mathbf{e}_r + r\dot{\phi}\,\mathbf{e}_\phi\right) \quad -(15)$$

where:

$$\mathcal{L} = -m(r)^{1/2} m c^2 \left(m(r) - \frac{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}{m(r) c^2}\right)^{1/2} + \frac{mMG}{r} \quad -(16)$$

$$= -m(r)^{1/2} m c^2 \left(m(r) - \frac{\dot{r}^2 + r^2 \dot{\phi}^2}{m(r) c^2}\right)^{1/2} + \frac{mMG}{r}$$

Therefore:

$$\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}} = \frac{\gamma m}{m(r)}\left(\dot{r}\,\mathbf{e}_r + r\dot{\phi}\,\mathbf{e}_\phi\right) \quad -(17)$$

$$\boxed{\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}} = \frac{\partial \mathcal{L}}{\partial \dot{r}}\mathbf{e}_r + \frac{1}{r}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\mathbf{e}_\phi} \quad -(18)$$

Similarly:

$$\boxed{\begin{aligned}\frac{\partial \mathcal{L}}{\partial \underline{r}} &= \frac{\partial \mathcal{L}}{\partial r} \underline{e}_r + \frac{1}{r} \frac{\partial \mathcal{L}}{\partial \phi} \underline{e}_\phi \\ &= \underline{\nabla} \mathcal{L}\end{aligned}} \quad -(19)$$

The Euler Lagrange equation is therefore:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \underline{\dot{r}}}\right) = \underline{\nabla} \mathcal{L} \quad -(20)$$

From eq. (16):

$$\underline{\nabla} \mathcal{L} = -\frac{mMG}{r^2} \underline{e}_r \quad -(21)$$

Therefore

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \underline{\dot{r}}}\right) = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}} \underline{e}_r\right) + \frac{d}{dt}\left(\frac{1}{r}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \underline{e}_\phi\right)$$

$$= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \underline{e}_r + \frac{\partial \mathcal{L}}{\partial \dot{r}} \frac{d \underline{e}_r}{dt} + \frac{d}{dt}\left(\frac{1}{r}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) \underline{e}_\phi + \frac{1}{r}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \frac{d \underline{e}_\phi}{dt} \quad -(22)$$

where:

$$\frac{d \underline{e}_r}{dt} = \dot{\phi}\, \underline{e}_\phi, \quad \frac{d \underline{e}_\phi}{dt} = -\dot{\phi}\, \underline{e}_r \quad -(23)$$

So

$$\boxed{\begin{aligned}\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \underline{\dot{r}}} &= \left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\dot{\phi}}{r}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) \underline{e}_r \\ &+ \left(\frac{\partial \mathcal{L}}{\partial \dot{r}} \dot{\phi} + \frac{d}{dt}\left(\frac{1}{r}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right)\right) \underline{e}_\phi\end{aligned}} \quad -(24)$$

) So using the Lagrangian:

$$\mathcal{L} = -m(r)^{1/2} m c^2 \left(m(r) - \frac{\dot{r}^2 + r^2\dot{\phi}^2}{m(r) c^2} \right)^{1/2} + \frac{mMG}{r} \quad -(25)$$

it is found that:

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\gamma}{m(r)^{1/2}} \dot{r}, \quad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\gamma}{m(r)^{1/2}} r^2 \dot{\phi} \quad -(26)$$

and:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\underline{r}}} = \frac{m}{m(r)^{1/2}} \left[\left(\frac{d}{dt}(\gamma \dot{r}) - \gamma r \dot{\phi}^2 \right) \underline{e}_r + \left(\frac{\partial \mathcal{L}}{\partial \dot{r}} \dot{\phi} + \frac{d}{dt}\left(\frac{1}{r} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) \right) \underline{e}_\phi \right] \quad -(27)$$

$$= \frac{m}{m(r)^{1/2}} \left(\frac{d\gamma}{dt} \dot{r} \underline{e}_r + \gamma\left(\ddot{r} - r\dot{\phi}^2 \right) \underline{e}_r + \gamma \dot{r} \dot{\phi} \underline{e}_\phi + \frac{d}{dt}\left(\frac{1}{r} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) \underline{e}_\phi \right)$$

Finally use:

$$\frac{d}{dt}\left(\frac{1}{r} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) = \frac{1}{m(r)^{1/2}} \frac{d}{dt}(\gamma r \dot{\phi}) \quad -(28)$$

$$= \frac{1}{m(r)^{1/2}} \left(\frac{d\gamma}{dt} r \dot{\phi} + \gamma\left(\dot{r}\dot{\phi} + r\ddot{\phi} \right) \right)$$

to find that:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\underline{r}}} = \frac{1}{m(r)^{1/2}} \left[\frac{d\gamma}{dt} \left(\dot{r} \underline{e}_r + r\dot{\phi} \underline{e}_\phi \right) + \gamma\left(\left(\ddot{r} - r\dot{\phi}^2 \right) \underline{e}_r + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi} \right) \underline{e}_\phi \right) \right] \quad -(29)$$

) which is eq. (28) of UFT 415(4) Q.E.D.

With the Lagrangian (25) the Newtonian and Lagrangian methods give exactly the same results.

The double cross checked equations of motion are therefore:

$$\frac{d\gamma}{dt}\dot{r} + \gamma\left(\ddot{r} - r\dot{\phi}^2\right) = -m(r)^{1/2}\frac{mG}{r}\left(\frac{1}{r} + \Omega r\right) \quad -(30)$$

and

$$\frac{d\gamma}{dt} r\dot{\phi} + r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0 \quad -(31)$$

The conserved angular momentum is:

$$L := \frac{\gamma m r^2}{m(r)}\dot{\phi} \quad -(32)$$

and Eq. (31) implies:

$$\frac{dL}{dt} = 0 \quad -(33)$$