412(6): Hamiltonian Method of Deriving the Spin Connection by Frame Rotation

Consider the classical Hamiltonian in the static frame:

$$H = \frac{1}{2} m v^2 + U \quad -(1)$$

Here U is the potential energy, and:

$$T = \frac{1}{2} m v^2 \quad -(2)$$

is the kinetic energy. The Hamiltonian is a constant of motion, so

$$\frac{dH}{dt} = 0 \quad -(3)$$

i.e.

$$\frac{d}{dt}\left(\frac{1}{2} m v^2 + U \right) = 0 \quad -(4)$$

so

$$m v \frac{dv}{dt} + \frac{dU}{dt} = 0 \quad -(5)$$

Now use:

$$\frac{dU}{dt} = \frac{dU}{dr}\frac{dr}{dt} = v \frac{dU}{dr} \quad -(6)$$

and it follows that:

$$F = m \frac{dv}{dt} = - \frac{dU}{dr} = - \frac{\partial U}{\partial r} \quad -(7)$$

if U depends only on r.

Therefore the force equation (7) follows as a direct consequence of the fact that H is a constant of motion, Q.E.D. Eq. (7) is also the Newtonian equivalence principle.

In the rotating frame:

$$H' = \frac{1}{2} m v'^2 + U \quad -(8)$$

because U is a scalar, independent of frame rotation.

) In the rotating frame:

$$\frac{dH'}{dt} = 0 \quad -(9)$$

from which it follows that

$$m v' \frac{dv'}{dt} + \frac{\partial U}{\partial r} = 0 \quad -(10)$$

i.e.

$$m \frac{dv'}{dt} = - \frac{\partial U}{\partial r} \quad -(11)$$

The total force in the rotated frame is, from Cartan geometry and the presence of torsion:

$$F' = -\left(\frac{\partial}{\partial r} - \Omega \right) U \quad -(12)$$

in which the covariant derivative is:

$$\frac{D}{\partial r} := \frac{\partial}{\partial r} - \Omega \quad -(13)$$

so the total force in the rotating frame is:

$$\underline{F}' = -\underline{\nabla} U + \underline{\Omega}' U \quad -(14)$$

in which the vacuum force is

$$\underline{F}'(\text{vac}) = \underline{\Omega}' U \quad -(14a)$$

The frame rotation

$$\phi' = \phi + \omega_1 t \quad -(15)$$

produces the spin connection and the vacuum force. The latter is ubiquitous and ever present. The spin connection is implied by Cartan geometry. Primed quantities are the observable quantities.

The rotating frame hamiltonian:

$$H' = \frac{1}{2} m v'^2 + U \quad -(16)$$

...duce the precessing orbit:

$$r = \frac{\alpha'}{1 + \epsilon' \cos\phi'} \quad -(17)$$

... in the preceding note and UFT papers and notes. The precession is

$$\Delta\phi = \omega_1 T \quad -(18)$$

Here ω_1 is the angular velocity ... the classical theory ... frame rotation and T is the time taken for one orbit of 2π radians. The orbital angular velocity ... the static frame is

$$\omega = \frac{2\pi}{T}. \quad -(19)$$

Therefore the constancy of the hamiltonian in the rotating frame produces the spin connection, which produces the vacuum force and the orbital precession.

It is remarkable that the simple frame rotation (15) produces so much information in such a simple way.

The following is a convenient summary of the theory. The primed quantities are those in the rotating frame, i.e. ... hypothesis, the frame rotation is due to torsion, i.e. spacetime torsion which is ubiquitous and ever present, and always takes place in this rotating frame. This physics ... static frame of Newton is an idealization. This is a consequence of ECE2 relativity.

The basic kinematic quantities are:

$$\underline{r}' = r \underline{e}_r \quad -(20)$$

because r does not change by hypothesis.

4) The velocity is:

$$\underline{v}' = \frac{d\underline{r}'}{dt} = \dot{r}\,\underline{e}_r' + r\dot{\phi}'\,\underline{e}_\phi' \quad -(21)$$

so

$$v'^2 = \dot{r}^2 + r^2\dot{\phi}'^2 \quad -(22)$$

The acceleration is:

$$\underline{a}' = \frac{d\underline{v}'}{dt} = (\ddot{r} - r\dot{\phi}'^2)\,\underline{e}_r' + (r\ddot{\phi}' + 2\dot{r}\dot{\phi}')\,\underline{e}_\phi' \quad -(23)$$

This is valid in general.

For a central force of type (10):

$$r\ddot{\phi}' + 2\dot{r}\dot{\phi}' = 0 \quad -(24)$$

so

$$\underline{F}' = m\underline{a}' = m(\ddot{r} - \dot{\phi}'^2)\,\underline{e}_r' \quad -(25)$$

It follows that

$$\ddot{r} - \dot{\phi}'^2 = -\frac{MG}{r} \quad -(26)$$

This is the Leibniz equation in the rotating frame.

The Lagrangian in the rotating frame is:

$$\mathcal{L}' = \frac{1}{2}mv'^2 + \frac{mMG}{r} \quad -(27)$$

and the Euler Lagrange equations are:

$$\frac{\partial \mathcal{L}'}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot{r}} \quad -(28)$$

and

$$\frac{\partial \mathcal{L}'}{\partial \phi'} = \frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot{\phi}'} \quad -(29)$$

Eq. (29) gives the constant angular momentum the rotating frame:

$$L' = m r^2 \dot{\phi}' \quad -(30)$$

$$= m r^2 \left(\omega + \omega_1 + t \frac{d\omega_1}{dt} \right)$$

Note that

$$\frac{dL'}{dt} = 0, \quad -(31)$$

because L' is a constant of motion.

From eq. (10), the force in the rotating frame is:

$$m \frac{d\underline{v}'}{dt} = - \underline{\nabla} U \quad -(32)$$

where

$$U = -\frac{m M G}{r} \quad -(33)$$

The total force in the rotating frame is:

$$\underline{F}' = - \underline{\nabla} U + \underline{\Omega}' U \quad -(34)$$

The spin connection introduces the vacuum force, which can be developed into a theory similar to that of the radiative corrections.

In general:

$$\underline{g}' = g_r' \underline{e}_r' + g_\phi' \underline{e}_\phi' \quad -(35)$$

and

$$\underline{\Omega}' = \Omega_r' \underline{e}_r' + \Omega_\phi' \underline{e}_\phi' \quad -(36)$$

and for dynamics in general the following two equations must be solved simultaneously:

o)

$$g_r' = \ddot{r} - r\dot{\phi}'^2 = -\frac{\partial \Phi}{\partial r} + \Omega_r' \Phi \quad -(37)$$

and

$$g_{\phi'}' = r\ddot{\phi}' + 2\dot{r}\dot{\phi}' = -\frac{1}{r}\frac{\partial \Phi}{\partial \phi'} + \Omega_\phi' \Phi \quad -(38)$$

In general:

$$\Phi = \Phi(r, \phi') \quad -(36)$$

In orbital theory:

$$\ddot{r} - r\dot{\phi}'^2 = -\frac{mG}{r^2} \quad -(37)$$

and

$$r\ddot{\phi}' + 2\dot{r}\dot{\phi}' = 0. \quad -(38)$$

These must be solved simultaneously. Here:

$$\dot{\phi}' = \frac{d\phi'}{dt} = \frac{d\phi}{dt} + \frac{d(\omega_1 t)}{dt}$$

$$= \omega + \omega_1 + t\frac{d\omega_1}{dt} \quad -(39)$$

and

$$\ddot{\phi}' = \frac{d}{dt}\dot{\phi}' \quad -(40)$$

$$= \frac{d\omega}{dt} + \frac{d\omega_1}{dt} + \frac{d\omega_1}{dt} + t\frac{d^2\omega_1}{dt^2}$$

$$= \frac{d\omega}{dt} + 2\frac{d\omega_1}{dt} + t\frac{d^2\omega_1}{dt^2}$$

The Newtonian theory is recovered for:

$$\omega_1 = 0 \quad -(41)$$
