412(4): Link Between ω_1 and the Spin Connection.

Consider the velocity in the static frame:

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\phi}\,\underline{e}_\phi. \quad -(1)$$

In the rotating frame:

$$\underline{v}' = \dot{r}\,\underline{e}_r' + r\dot{\phi}'\,\underline{e}_\phi' \quad -(2)$$

where

$$\phi' = \phi + \omega_1 t \quad -(3)$$

and

$$\dot{\phi}' = \dot{\phi} + t\frac{d\omega_1}{dt} + \omega_1 \quad -(4)$$

$$= \omega + \omega_1 + t\frac{d\omega_1}{dt}$$

In the static frame the unit vectors are:

$$\underline{e}_r = \underline{i}\cos\phi + \underline{j}\sin\phi \quad -(5)$$

$$\underline{e}_\phi = -\underline{i}\sin\phi + \underline{j}\cos\phi \quad -(6)$$

and in the rotating frame:

$$\underline{e}_r' = \underline{i}\cos(\phi + \omega_1 t) + \underline{j}\sin(\phi + \omega_1 t) \quad -(7)$$

$$\underline{e}_\phi' = -\underline{i}\sin(\phi + \omega_1 t) + \underline{j}\cos(\phi + \omega_1 t) \quad -(8)$$

The acceleration in the rotating frame is:

$$\underline{a}' = (\ddot{r} - r\dot{\phi}'^2)\,\underline{e}_r' + (r\ddot{\phi}' + 2\dot{r}\dot{\phi})\,\underline{e}_\phi' \quad -(9)$$

By hypothesis, the rotation of the frame produces the force:

$$\underline{F}' = m\underline{a}' = -\frac{\partial U}{\partial r} + \underline{\Omega}' U \quad -(10)$$

where $\underline{\Omega}$ is the spin connection, So:

$$\underline{F}' = -\frac{mMG}{r^2}\underline{e}_r' + \underline{\Omega}'U \quad -(11)$$

here the gravitational potential energy is:

$$U = -\frac{mMG}{r} \quad -(12)$$

In general:

$$\underline{\Omega}' = \Omega_r'\underline{e}_r' + \Omega_\phi'\underline{e}_\phi' \quad -(13)$$

Assume for simplicity that:

$$\Omega_\phi' = 0 \quad -(14)$$

then:

$$(\ddot{r} - r\dot{\phi}'^2) = -\frac{mMG}{r^2} + \Omega_r' \quad -(15)$$

i.e.

$$\ddot{r} - r\left(\omega + \omega_1 + t\frac{d\omega_1}{dt}\right)^2 = -\frac{mMG}{r^2} + \Omega_r' \quad -(16)$$

where the angular momentum is:

$$L = mr^2\omega \quad -(17)$$

is the angular momentum, a constant of motion.

Therefore eq. (16) is the link between ω_1 and Ω_r'.
