412(5): Effect of Frame Rotation on Newtonian Orbital Theory

By hypothesis, frame rotation always exists in the universe because of spacetime torsion. Therefore the entire structure of Newtonian theory is changed. One of the major consequences of frame rotation is that orbits precess. The constants of motion of the Newtonian theory are the hamiltonian and the angular momentum, and these are changed to H' and L', constants of motion in the rotating frame. In consequence the conic section orbit in the rotating frame is changed to:

$$r = \frac{\alpha'}{1+\epsilon'\cos\phi'} \quad -(1)$$

where

$$\phi' = \phi + \omega_1 t \quad -(2)$$

By hypothesis of the de Sitter rotation (2), r remains the same in the rotated and unrotated frames. The angular momentum in the rotated frame is:

$$L' = \mu r^2 \frac{d\phi'}{dt} \quad -(3)$$

and the hamiltonian in the rotated frame is described in note 412(3):

$$H' = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\frac{L'}{mr^2} - \frac{mMG}{r} \quad -(4)$$

Here

$$\mu = \frac{mM}{m+M} \quad -(5)$$

is the reduced mass.

Denoting:

$$k = mMG \quad -(6)$$

the half right latitude α' and the eccentricity ϵ' in the rotating frame are constants of motion defined by:

$$\alpha' = \frac{L'^2}{\mu k} \quad -(7)$$

d

$$\epsilon' = \left(1 + \frac{2H'L'^2}{\mu k^2}\right)^{1/2} \quad -(8)$$

The Hamiltonian may be expressed as:

$$H' = -\frac{mMG}{2a'} \quad -(9)$$

so the semi major axis a' in the rotating frame is also a constant of motion. For planetary motion in the rotating frame:

$$a' = \frac{\alpha'}{1-\epsilon'^2} \quad -(10)$$

$$= \frac{k}{2|H'|}$$

The semi minor axis b' is also a constant of motion:

$$b' = \frac{\alpha'}{(1-\epsilon'^2)^{1/2}} = \frac{L'}{(2\mu|H'|)^{1/2}} \quad -(11)$$

Here

$$|H'| = \frac{mMG}{2a'} \quad -(12)$$

the modulus or absolute value of the Hamiltonian in the rotating frame.

Note that:

$$\frac{L'}{mr'^2} = \frac{d\phi'}{dt} = \omega + \omega_1 + t\frac{d\omega_1}{dt} \quad -(13)$$

where ω is the angular velocity of the orbit in the static frame and ω_1 is the angular velocity of frame rotation.

So:
$$L' = mr'^2 \left(\omega + \omega_1 + t \frac{d\omega_1}{dt} \right) \quad -(14)$$
$$= \text{constant}$$

If
$$\frac{d\omega_1}{dt} \neq 0 \quad -(15)$$
the orbital radius, r, decreases with increasing t in order to keep L' constant.

The orbit precesses and shrinks in the rotating frame, and this is observed in the Hulse Taylor binary pulsar. This is direct astronomical proof of the existence of a rotating frame and spacetime torsion.

Therefore an expression is needed for the time t in Eq. (14). This is found from:
$$\frac{d\phi'}{dt} = \frac{L'}{mr^2} = \frac{L'}{md'^2} \left(1 + \epsilon' \cos\phi'\right)^2 \quad -(16)$$

so
$$t = \frac{md'^2}{L'^2} \int \frac{d\phi'}{\left(1+\epsilon'\cos\phi'\right)^2} \quad -(17)$$

The integral is evaluated as follows:
$$\int \frac{d\phi'}{\left(1+\epsilon'\cos\phi'\right)^2} = \frac{\epsilon \sin\phi'}{\left(\epsilon'^2 - 1\right)\left(1+\epsilon'\cos\phi'\right)} - \frac{1}{\left(\epsilon'^2-1\right)} \int \frac{d\phi'}{1+\epsilon'\cos\phi'} \quad -(18)$$

in which:

$$\int \frac{d\phi'}{1+\epsilon'\cos\phi'} = \frac{2}{(1-\epsilon'^2)^{1/2}} \tan^{-1}\left[\frac{(1-\epsilon')\tan\phi'/2}{(1-\epsilon'^2)^{1/2}}\right] \quad -(19)$$

in which

$$\phi' = \phi + \omega_1 t \quad -(20)$$

In general, eq. (17) must be solved for t. This is a complicated process needing a supercomputer and specialist methods.

However, the problem is greatly simplified if the time T needed for one orbit of 2π is considered:

$$T' = \frac{m\alpha^2}{L'} \int_0^{2\pi} \frac{d\phi'}{(1+\epsilon'\cos\phi')^2} \quad -(21)$$

In this case we need only consider $\phi' = 2\pi$ and $\phi' = 0$, and by Kepler's third law:

$$T'^2 = \frac{4\pi^2\mu}{mMG} a'^3 \quad -(22)$$

$$= \left(\frac{m\alpha'^2}{L'} \int_0^{2\pi} \frac{d\phi'}{(1+\epsilon'\cos\phi')^2}\right)^2$$

and

$$L' = m r^2 \left(\omega + \omega_1 + T'\frac{d\omega_1}{dt}\right) \quad -(23)$$

It can be assumed that

$$\mu \sim m \sim \frac{mM}{m+M} \quad -(24)$$

if

$$M \gg m \quad -(25)$$

) It can be seen from Kepler's third law, eq. (22),
at T' is a constant of motion, because a' is a
artant of motion. Here T' is the time taken to
omplete one orbit of 2π radians.

In the rotating frame the orbit precesses according
to:

$$\Delta\phi' = \omega_1 T' \quad - (26)$$

so after T' has elapsed, the orbit advances by
$\Delta\phi'$. As shown in previous notes, eq. (26) is the
classical limit of a relativistic theory.