411(3): Proof of Precession and Shrinkage

Consider the orbit:

$$r = \frac{\alpha'}{1+\epsilon' \cos(\phi+\omega t)} \quad -(1)$$

where

$$\omega = \frac{d\phi}{dt} \quad -(2)$$

is the angular velocity of the rotating frame.

In one orbit the argument of the cosine increases by 2π, so:

$$\phi + \omega t = 2\pi \quad -(3)$$

and

$$\phi = 2\pi - \omega t \quad -(4)$$

So there is a displacement:

$$\Delta = -\omega t \quad -(5)$$

every orbit and the orbit precesses, Q.E.D. Let the time taken for one orbit be T', then the displacement is

$$\Delta = -\omega T' \quad -(6)$$

By Kepler's third law:

$$T'^2 = \left(\frac{4\pi^2}{mG}\right) a'^3 \quad -(7)$$

where

$$a' = \frac{\alpha'}{1-\epsilon'^2} \quad -(8)$$

is the semi major axis.

By definition:

$$L' = mr^2\left(2\omega + t\frac{d\omega}{dt}\right) \quad -(9)$$

So the orbital radius is:

2) and
$$r^2 = \frac{L'}{m}\left(2\omega + t\frac{d\omega}{dt}\right)^{-1} \quad -(10)$$

nd as t increases,
$$\boxed{r \to 0} \quad -(11)$$

and the orbit precesses and shrinks, Q.E.D.

This theory is based on eq (24) of the preceding Note 411(2):

$$\omega' = \frac{d\phi'}{dt} = \frac{d\phi}{dt} + \frac{d}{dt}(\omega t) \quad -(12)$$
$$= 2\omega + t\frac{d\omega}{dt}$$

in which it has been assumed that the angular velocity of frame rotation is the same as the angular velocity of the orbit (ω). More generally the angular velocity of frame rotation, denoted ω_1, is not the same as ω. So:

$$\omega' = \frac{d\phi'}{dt} = \frac{d\phi}{dt} + \frac{d}{dt}(\omega_1 t) \quad -(13)$$
$$= \frac{d\phi}{dt} + \omega_1 + t\frac{d\omega_1}{dt}$$
$$= \omega + \omega_1 + t\frac{d\omega_1}{dt}$$

In this case
$$L' = \mu r^2\left(\omega + \omega_1 + t\frac{d\omega_1}{dt}\right) \quad -(14)$$
$$= \text{constant of motion.}$$

The precessing orbit is now:

$$r = \frac{\alpha'}{1+\epsilon'\cos\phi'} \quad -(15)$$

here

$$\phi' = \phi + \omega_1 t \quad -(16)$$

The precession per orbit is given by:

$$\boxed{\Delta = -\omega_1 T'} \quad -(17)$$

and the shrinkage is given by:

$$\boxed{r^2 = \frac{L'}{m\left(\omega+\omega_1+t\frac{d\omega_1}{dt}\right)}} \quad -(18)$$

Precessions in the solar system are very small,

so

$$\omega \gg \omega_1 \quad -(19)$$

and the shrinkage is very small in the solar system. In the Hulse Taylor binary pulsar the shrinkage is observable.

So eq. (18) should be compared with data from the Hulse Taylor binary pulsar