ll (2): The Effect of Frame Rotation on Orbital Theory

The rotation is defined by replacing ϕ of the plane polar system with ϕ', where:

$$\phi' = \phi + \omega t \quad - (1)$$

where ω is the angular velocity of the frame rotation and t the time. The unit vectors of the rotating frame are defined by:

$$\underline{e}_r' = \underline{i}\cos\phi' + \underline{j}\sin\phi' \quad - (2)$$

$$\underline{e}_\phi' = -\underline{i}\sin\phi' + \underline{j}\cos\phi' \quad - (3)$$

where $\underline{i}$ and $\underline{j}$ are the Cartesian unit vectors fixed in space. The relation between unit vectors are:

$$\frac{d\underline{e}_r'}{d\phi'} = \underline{e}_\phi' \quad - (4)$$

and

$$\frac{d\underline{e}_\phi'}{d\phi'} = -\underline{e}_r' \quad - (5)$$

In general:

$$\underline{r}' = r'\underline{e}_r' \quad - (6)$$

However, in considering precession theory the infinitesimal line element is:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2$$
$$= dt^2(c^2 - v_N^2) \quad - (7)$$

in which the Newtonian velocity is:

$$v_N^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2 \quad - (8)$$

and the precession theory is developed with eq. (1) and a fixed r, i.e. r is assumed to be unchanged by eq. (1), so

$$r' = r \quad - (9)$$

It follows that:

$$\underline{r}' = r\underline{e}_r' \quad - (10)$$

) So the linear velocity in the rotating frame defined by eq.(1) is:

$$\underline{v}' = \frac{d\underline{r}'}{dt} = \frac{dr}{dt}\underline{e}_r' + r\frac{d\underline{e}_r'}{dt} \quad -(11)$$

From eqs. (4) and (5):

$$\frac{d\underline{e}_r'}{dt} = \underline{e}_\phi' \frac{d\phi'}{dt} \quad -(12)$$

and

$$\frac{d\underline{e}_\phi'}{dt} = -\underline{e}_r' \frac{d\phi'}{dt} \quad -(13)$$

So

$$\underline{v}' = \frac{dr}{dt}\underline{e}_r' + r\dot{\phi}'\underline{e}_\phi' \quad -(14)$$

$$= \dot{r}\,\underline{e}_r' + r\dot{\phi}'\,\underline{e}_\phi'$$

So

$$v'^2 = \dot{r}^2 + r^2\dot{\phi}'^2 \quad -(15)$$

$$= \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi'}{dt}\right)^2$$

in which:

$$d\phi' = d\phi + \omega\, dt \quad -(16)$$

The Lagrangian is

$$\mathcal{L}' = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}'^2\right) - U(r) \quad -(17)$$

where μ is the reduced mass and $U(r)$ the gravitational potential energy:

$$U = -\frac{mMG}{r} \quad -(18)$$

The Euler Lagrange equations of the rotating frame are:

3)
$$\frac{\partial \mathcal{L}'}{\partial \phi'} = \frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot{\phi}'} \quad -(19)$$

and
$$\frac{\partial \mathcal{L}'}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot{r}} \quad -(20)$$

Eq. (19) gives the conserved angular momentum:
$$L' = \mu r^2 \dot{\phi}' \quad -(21)$$

and eq. (20) gives
$$\mu(\ddot{r} - r\dot{\phi}'^2) = -\frac{\partial U}{\partial r} = F(r) \quad -(22)$$

which is equivalent to the Binet equation of the rotating frame:
$$\frac{d^2}{d\phi'^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{L'^2} F(r) \quad -(23)$$

The angular velocity in the rotating frame is:
$$\omega' = \frac{d\phi'}{dt} = \frac{d\phi}{dt} + \frac{d}{dt}(\omega t) \quad -(24)$$
$$= 2\omega + t\frac{d\omega}{dt}$$

So for a non-zero angular acceleration $d\omega/dt$ the rotating frame angular velocity ω' increases with time. The angular momentum in the rotating frame is a constant of motion:
$$L' = \mu r^2 \omega' = \text{constant} \quad -(25)$$

4) So:

$$L' = \mu r^2 \left(2\omega + t \frac{d\omega}{dt} \right) = \text{constant} \quad -(26)$$

So r decreases with time because ω' increases with time.

This is exactly what is observed in the orbit of a Hulse Taylor binary pulsar. The precession of the pulsar is defined by eq. (1) and the universal laws of precession.

The hamiltonian in the rotating frame is given by:

$$H' = \frac{1}{2} \mu \dot{r}^2 + \frac{1}{2} \frac{L'^2}{mr^2} + U(r) \quad -(27)$$

and as in Marion and Thornton section 7.4 of the third edition leads to the conic section orbit:

$$r = \frac{\alpha'}{1 + \epsilon' \cos \phi'} \quad -(28)$$

where the half right latitude is:

$$\alpha' = \frac{L'^2}{\mu m M G} \quad -(29)$$

and the eccentricity is:

$$\epsilon' = \left(1 + \frac{2H'L'^2}{\mu (mMG)^2} \right)^{1/2} \quad -(30)$$

Therefore rotating the plane polar frame of reference using eq. (1) produces the orbit.

5)

$$\boxed{r = \frac{\alpha'}{1+\epsilon' \cos(\phi + \omega t)}} \quad -(31)$$

and this is a precessing orbit defined by the universal law of precession obtained from eq. (1).

The Newtonian velocity in the rotating frame is given by:

$$v_N^{2\prime} = \frac{mMG}{\mu}\left(\frac{2}{r} - \frac{1}{a'}\right) \quad -(32)$$

where a' is the semi major axis in the rotating frame

$$a' = \frac{\alpha'}{1-\epsilon'^2} = \frac{mMG}{2|H'|} \quad -(33)$$

The Newtonian velocity in the rotating frame is also given by:

$$v_N^{2\prime} = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi'}{dt}\right)^2$$

$$= \left(\frac{dr}{dt}\right)^2 + r^2\left(2\omega + t\frac{d\omega}{dt}\right)^2 \quad -(34)$$

This increases with t so as the orbit shrinks the orbital velocity increases given a finite angular acceleration $d\omega / dt$.

So the simple frame rotation (1) gives both the universal law of precession and the main features of the Hulse Taylor binary pulsar.