09(3): Thomas Rotation at the Newtonian Velocity $v_N$.

This procedure results in the Thomas precession:

$$\Delta \phi_T = 2\pi(\gamma - 1) := 2\pi\beta \quad -(1)$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = \frac{dt}{d\tau} \quad -(2)$$

the Lorentz factor and where $v$ is the Newtonian velocity. The relativistic kinetic energy is given immediately from eq. (1) as

$$T = (\gamma - 1)mc^2 = \beta mc^2 \quad -(3)$$

and the total energy as:

$$E = T + mc^2 = \gamma mc^2 = (1+\beta)mc^2 \quad -(4)$$

The Lorentz factor $\gamma$ can therefore be derived from the Thomas factor $\beta$, and from a Thomas rotation. All the fundamental quantities of ECE2 covariant physics can be derived from a Thomas rotation.

The usual way of deriving the Lorentz factor is to consider the invariance under four rotation:

$$x^\mu x_\mu = x^{\mu'} x_\mu' \quad -(5)$$

where

$$x^\mu = (ct, X, Y, Z) \quad -(6)$$

$$x^{\mu'} = (ct', X', Y', Z') \quad -(7)$$

$$c^2t^2 - (X^2 + Y^2 + Z^2) = c^2t'^2 - (X'^2 + Y'^2 + Z'^2) \quad -(8)$$

Applying eq. (8) to infinitesimals:

$$c^2 dt^2 - dr^2 = c^2 dt'^2 - dr'^2 \quad -(9)$$

where

$$dr^2 = dX^2 + dY^2 + dZ^2 \quad -(10)$$

$$dr'^2 = dX'^2 + dY'^2 + dZ'^2 \quad -(11)$$

The moving frame $(ct', x', y', z')$ is the frame in
which a particle is at rest, so its velocity in this frame is
zero and $dx'^2 + dy'^2 + dz'^2 = v'^2 dt'^2 = 0$ – (12)

$$dX'^2 + dY'^2 + dZ'^2 = v'^2 dt'^2 = 0 \quad - (12)$$

$$dx^2 + dy^2 + dz^2 = v^2 dt^2 \quad - (13)$$

giving:

Eq. (9) becomes:

$$(c^2 - v^2) dt^2 = c^2 dt'^2 := c^2 d\tau^2 \quad - (14)$$

where $d\tau$ is the infinitesimal of proper time, the time
in the moving frame.

From eq. (14) the Lorentz factor is:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = \frac{dt}{d\tau} \quad - (15)$$

and is therefore derived from the four rotation of $x^\mu$:
In plane polar coordinates the result of the four rotation
is:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 \quad - (16)$$

The relativistic kinetic energy is calculated from the
work done:

$$W_{12} = \int_1^2 \underline{F} \cdot d\underline{r} = T_2 - T_1 \quad - (17)$$

where the relativistic force is:

$$\underline{F} = \frac{d}{dt}(\gamma m \underline{v}) \quad - (18)$$

So:

$$T = W = \int \frac{d}{dt}(\gamma m \underline{v}) \cdot \underline{v}\, dt \quad - (19)$$

in which

$$d\underline{r} = \underline{v}\, dt \quad - (20)$$

It follows that:

$$T = m\int_0^v v\, d(\gamma v) \quad -(21)$$

$$= \gamma m v^2 - m\int_0^v \gamma v\, dv$$

$$= \gamma m v^2 + mc^2\left(1 - v^2/c^2\right)^{1/2} - mc^2$$

$$= (\gamma - 1)mc^2.$$

So:

$$T = W = \int \underline{F}\cdot\underline{v}\, dt = (\gamma-1)mc^2 = \beta mc^2 \quad -(22)$$

where

$$\beta = \frac{\Delta\phi}{2\pi} \quad -(23)$$

The total energy is given by:

$$E = T + mc^2 \quad -(24)$$

$$= \gamma mc^2 = (1+\beta)mc^2$$

The two methods of deriving the relativistic kinetic energy are similar, because both depend on rotations, but the conventional method requires additional concepts, notably the work done. From eq. (22), the Thomas rotation produces the same work:

$$\beta = \frac{1}{mc^2}\int \underline{F}\cdot\underline{v}\, dt = \gamma - 1 \quad -(25)$$

The four rotation (5) to (7) in plane polar coordinates is:

$$c^2dt^2 - dr^2 - r^2d\phi^2 = c^2dt'^2 - dr'^2 - r'^2d\phi'^2 \quad -(26)$$

In the moving frame:

$$dr'^2 + r'^2 d\phi'^2 = 0 \quad -(27)$$

so

$$c^2dt^2 - dr^2 - r^2d\phi^2 = c^2dt'^2 = c^2d\tau^2 \quad -(28)$$

with

$$v^2 dt^2 = dr^2 + r^2 d\phi^2 \quad -(29)$$

so

$$(c^2 - v^2) dt^2 = c^2 d\tau^2 \quad -(30)$$

which is equivalent to a Lorentz boost.

The infinitesimal line element in plane polar coordinates is derived from:

$$ds^2 = \underline{dr} \cdot \underline{dr} \quad -(31)$$

where

$$\underline{r} = X\underline{i} + Y\underline{j} \quad -(32)$$

("Vector Analysis Problem Solver", p. 1031). So:

$$\underline{r} = \underline{i}\, r\cos\phi + \underline{j}\, r\sin\phi \quad -(33)$$

and

$$\underline{dr} = \frac{\partial \underline{r}}{\partial r} dr + \frac{\partial \underline{r}}{\partial \phi} d\phi \quad -(34)$$

where

$$\frac{\partial \underline{r}}{\partial r} = \underline{i} \cos\phi + \underline{j} \sin\phi \quad -(35)$$

and

$$\frac{\partial \underline{r}}{\partial \phi} = -\underline{i}\, r\sin\phi + \underline{j}\, r\cos\phi \quad -(36)$$

so

$$ds^2 = dr^2 + r^2 d\phi^2 \quad -(37)$$

and

$$\underline{dr} = (\cos\phi\, dr - r\sin\phi\, d\phi)\,\underline{i} + (\sin\phi\, dr + r\cos\phi\, d\phi)\,\underline{j} \quad -(38)$$

The derivation of eq. (37) starts from eq. (32). The low rotation produces:

$$\begin{aligned}\underline{r}' &= X'\underline{i} + Y'\underline{j} \\ &= r\cos(\phi + \omega t)\,\underline{i} + r\sin(\phi + \omega t)\,\underline{j}\end{aligned} \quad -(39)$$

2) So the Thomas rotation is a four rotation defined by:

$$x^\mu = (ct, X, Y) = (ct, r\cos\phi, r\sin\phi) \quad -(40)$$

and

$$x^{\mu'} = (ct, X', Y') = (ct, r\cos(\phi+\omega t), r\sin(\phi+\omega t)) \quad -(41)$$

and

$$x^\mu x_\mu = x^{\mu'} x_{\mu'} \quad -(42)$$

i.e.

$$c^2t^2 - (X^2+Y^2) = c^2t'^2 - (X'^2+Y'^2). \quad -(43)$$

Eq. (43) means:

$$X'^2 + Y'^2 = X^2 + Y^2 = r^2 \quad -(44)$$

because:

$$\cos^2\phi + \sin^2\phi = \cos^2(\phi+\omega t) + \sin^2(\phi+\omega t) = 1 \quad -(45)$$

Therefore the Thomas rotation is a type of four rotation. It changes the infinitesimal line element to

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2(d\phi + \omega\, dt)^2 \quad -(46)$$

in which the Thomas velocity is:

$$v = \omega r \quad -(47)$$

As in Note 405(5), eq. (46) can be written as: $\quad -(47)$

$$ds^2 = \left(1 - \frac{v^2}{c^2}\right)\left(c^2 dt^2 - 2r^2\Omega\, d\phi\, dt\right) - (dr^2 + r^2 d\phi^2)$$

where

$$\Omega = \omega\left(1 - \frac{v^2}{c^2}\right)^{-1} \quad -(48)$$

It follows that the infinitesimal of time is changed by Thomas rotation to:

$$dt' = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} dt. \quad -(49)$$

The Thomas precession is defined by:

$$\Delta \phi = \Omega\, dt' - \omega\, dt \quad -(50)$$

$$= 2\pi(\gamma - 1)$$

$$= 2\pi\beta$$

Q.E.D., and the relativistic kinetic energy is:

$$T = mc^2\beta \quad -(51)$$

The four rotations (8) and (43) both produce the relativistic kinetic energy (51). The Lorentz boost described by Eq. (8) is equivalent to the Thomas rotation (43), so the latter is the most fundamental concept of relativity.