408(1): Energy Levels of the Dirac Atom in Terms of the Thomas Half

Consider the Einstein energy equation:

$$E^2 = p^2c^2 + m^2c^4 \quad -(1)$$

$$= \gamma^2 m^2 c^4,$$

which is a limit of the ECE wave equation. Here γ is the Lorentz factor defined by:

$$\gamma = \left(1 - \frac{v_N^2}{c^2}\right)^{-1/2} \quad -(2)$$

The Thomas precession per radian is defined by:

$$\frac{\Delta\phi_T}{2\pi} = \gamma - 1 \quad -(3)$$

so

$$\gamma = 1 + \frac{\Delta\phi_T}{2\pi} \quad -(4)$$

and the Einstein energy equation is:

$$E^2 = \left(1 + \frac{\Delta\phi_T}{2\pi}\right)^2 m^2c^4 \quad -(5)$$

$$= p^2c^2 + m^2c^4.$$

The relativistic total energy is:

$$E = \gamma mc^2 = \left(1 + \frac{\Delta\phi_T}{2\pi}\right)mc^2 = \hbar\omega \quad -(6)$$

So relativistic energy levels in general are given by:

$$E = \hbar\omega = \left(1 + \frac{\Delta\phi_T}{2\pi}\right)mc^2 \quad -(7)$$

So relativistic energy levels are Thomas precessions, for example in the Dirac and Sommerfeld atoms.

2) using:

$$H = E + U \ldots - (8)$$

where H is the Hamiltonian and U is the potential energy, eq. (1) becomes:

$$(H-U)^2 = p^2c^2 + m^2c^4 = (H-U)(H+U) - (9)$$

So

$$(H-U)^2 - m^2c^4 = p^2c^2 - (10)$$

$$= (H - U - mc^2)(H - U + mc^2) = p^2c^2 - (11)$$

So

$$E = \frac{p^2c^2}{H - U + mc^2} + mc^2 - (12)$$

This equation has been derived and developed in several UFT papers.

Dirac assumed that:

$$U << H - (13)$$

so

$$H \sim E - (14)$$

In the limit

$$v << c - (15)$$

$$E = \gamma mc^2 \rightarrow mc^2 - (16)$$

So the Dirac approximation is:

$$E = \frac{p^2c^2}{2mc^2 - U} + mc^2 - (17)$$

i.e.

$$T = (\gamma - 1)mc^2 = \frac{\Delta\phi_T}{2\pi}mc^2 = \frac{p^2}{2m}\left(1 + \frac{U}{2mc^2}\right) - (18)$$

where we have used:

$$E - mc^2 = \left(1 - \frac{U^2}{2mc^2}\right)^{-1} \frac{p^2}{2m} \sim \frac{p^2}{2m}\left(1 + \frac{U^2}{2mc^2}\right) \quad -(19)$$

In the low velocity limit:

$$\frac{\Delta\phi_T}{2\pi} \sim \frac{1}{2}\frac{v^2}{c^2} \quad -(20)$$

From eq. (18):

$$\frac{\Delta\phi_T}{2\pi} = \frac{p^2}{2m^2c^2}\left(1 + \frac{U}{2mc^2}\right) \quad -(21)$$

$$= \frac{\gamma^2 v^2}{2c^2}\left(1 + \frac{U}{2mc^2}\right)$$

So the relativistic version of the Thomas half in this Dirac approximation is

$$\boxed{\frac{\Delta\phi_T}{2\pi} = \frac{1}{2}\frac{v^2}{c^2}\gamma^2\left(1 + \frac{U}{2mc^2}\right)} \quad -(22)$$

The energy levels of the Dirac atom can be expressed as

$$T = (\gamma - 1)mc^2 = \frac{\Delta\phi_T}{2\pi} mc^2 = \frac{p^2}{2m}\left(1 + \frac{U}{2mc^2}\right) \quad -(23)$$

In the low velocity approximation these can be quantized with the Schroedinger quantization:

$$p^2\phi = -\hbar^2\nabla^2\phi \quad -(24)$$

So

$$\langle T \rangle = \left\langle \frac{p^2}{2m} \right\rangle + \left\langle \frac{p^2 U}{4m^2c^2} \right\rangle \quad -(25)$$

It follows that for the H atom:

$$\langle T \rangle = \frac{1}{2}\frac{\alpha^2}{n^2}mc^2 - \frac{1}{4m^2c^2}\int \psi^* \nabla^2 U \psi \quad -(26)$$

$$= \left\langle \frac{\Delta \phi_T}{2\pi} \right\rangle mc^2$$

The same result for the Schroedinger H atom is:

$$\langle T \rangle = \frac{1}{2}\frac{\alpha^2}{n^2}mc^2 \quad -(27)$$

so the relativistic correction is the second term in eq. (26).

The SU(2) basis is used in Eq. (21) to obtain the spin orbit structure of H:

$$\frac{\Delta \phi_T}{2\pi} = \frac{1}{2m^2c^2}\,\underline{\sigma}\cdot\underline{p}\left(1 + \frac{U}{2mc^2}\right)\underline{\sigma}\cdot\underline{p} \quad -(28)$$

and the potential energy is added to eq. (26) to obtain:

$$\langle H \rangle = \frac{1}{2}\frac{\alpha^2}{n^2}mc^2 + \langle U \rangle + \frac{1}{m^2c^2}\left\langle \underline{\sigma}\cdot\underline{p}\, U\, \underline{\sigma}\cdot\underline{p} \right\rangle \quad -(29)$$

For a Coulombic potential:

$$\langle U \rangle = -\frac{\alpha^2}{n^2}mc^2 \quad -(30)$$

so

$$\langle H \rangle = -\frac{1}{2}\left(\frac{\alpha}{n}\right)^2 mc^2 - \frac{\hbar^2}{4m^2c^2}\left\langle \underline{\sigma}\cdot\underline{\nabla}\left(U\,\underline{\sigma}\cdot\underline{p}\right)\right\rangle \quad -(31)$$

For the Schroedinger H atom:

$$\langle H \rangle = \langle T \rangle + \langle U \rangle \quad -(32)$$

$$= -\frac{1}{2}\left(\frac{\alpha}{n}\right)^2 mc^2$$

where

$$\frac{v}{c} = \frac{\alpha}{n} \quad -(33)$$

Here α is the fine structure constant.

It is seen that Sommerfeld atoms are due to the Thomas precession.

Several previous UFT papers have developed the Dirac approximation and the Dirac equation has been developed into the fermion equation.