408(3): The Fundamental Role of the Thomas Precession in the Exact Dirac Equation

The exact Dirac equation is obtained from the hamiltonian:

$$H = \gamma mc^2 + U = (m^2 p^2 + m^2 c^4)^{1/2} + U \quad -(1)$$

Here:

$$E = H - U = \gamma mc^2 \quad -(2)$$

is the total relativistic energy, and:

$$T = (\gamma - 1) mc^2 \quad -(3)$$

is the relativistic kinetic energy. The Thomas half is defined by:

$$\frac{\Delta \phi_T}{2\pi} = \gamma - 1. \quad -(4)$$

It is obtained from rotating the line element:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - v_N^2 dt^2 \quad -(5)$$

here

$$v_N^2 dt^2 = dX^2 + dY^2 + dZ^2 = dr^2 + r^2 d\phi^2 \quad -(6)$$

and

$$\gamma = \left(1 - \frac{v_N^2}{c^2}\right)^{-1/2} \quad -(7)$$

The rotation defined by:

$$d\phi' = d\phi + \omega dt \quad -(8)$$

and gives the Thomas precession:

$$\Delta \phi_T = 2\pi (\gamma - 1) \quad -(9)$$

From eq. (1):

$$E = H - U = \gamma mc^2 = (m^2 p^2 + m^2 c^4)^{1/2} \quad -(10)$$

so

$$(H - U)^2 - m^2 c^4 = m^2 p^2 \quad -(11)$$

i.e.

$$(E - mc^2)(E + mc^2) = m^2 p^2 \quad -(12)$$

and

$$E = \frac{p^2 c^2}{E + mc^2} + mc^2 \quad -(13)$$

i.e. $$T = E - mc^2 = \frac{p^2c^2}{E+mc^2} = \frac{p^2}{m(\gamma+1)} \quad -(14)$$

t follows that $$T \xrightarrow[\gamma\to 1]{} \frac{p^2}{2m} \quad -(15)$$

which is the classical kinetic energy.

Eq. (14) is a rearrangement of the classical hamiltonian and is referred to as the exact result.

The Dirac approximation is:

$$H \sim mc^2, \quad U \ll H \quad -(16)$$

so $$E - mc^2 = \frac{p^2c^2}{2mc^2 - U} = \frac{p^2}{2m}\left(1 - \frac{U}{2mc^2}\right)^{-1} \quad -(17)$$

so the relevant hamiltonian is:

$$H_0 = H - mc^2 = E - mc^2 + U \quad -(18)$$

$$= \frac{p^2}{2m}\left(1 - \frac{U}{2mc^2}\right)^{-1} + U$$

$$\sim \frac{p^2}{2m} + U + \frac{\underline{\sigma}\cdot\underline{p}\, U \underline{\sigma}\cdot\underline{p}}{4m^2c^2} \quad -(19)$$

in the SU(2) basis.

The Dirac quantization is:

$$p^2\psi = -\hbar^2\nabla^2\psi \quad -(20)$$

so $$H_0\psi = \left(-\frac{\hbar^2\nabla^2}{2m} + U\right)\psi - \frac{i\hbar}{4m^2c^2}\underline{\sigma}\cdot\underline{\nabla}\left(U\underline{\sigma}\cdot\underline{p}\psi\right) \quad -(21)$$

If ψ are assumed to be the non-relativistic

) wave functions, then the energy levels of the H atom are
altered by a spin orbit term, giving excellent agreement
with experimental data.

The energy levels of the Schroedinger H atom are:

$$H_0 = \langle H_0 \rangle = \int \psi^* H_0 \psi \, d\tau$$

$$= -\frac{\hbar^2}{2m} \int \psi^* \nabla^2 \psi \, d\tau + \int \psi^* U \psi \, d\tau \quad -(22)$$

$$= \frac{1}{2} \frac{\alpha^2}{n^2} mc^2 - \frac{\alpha^2}{n^2} mc^2$$

$$= -\frac{1}{2} \frac{\alpha^2}{n^2} mc^2 .$$

Here α is the fine structure constant and n is the principal
quantum number. In eq. (22), as shown in UFT207:

$$\frac{v}{c} = \frac{\langle v \rangle}{c} = \frac{\alpha}{n} \quad -(23)$$

and

$$\frac{\Delta \phi_T}{2\pi} \xrightarrow[v \ll c]{} \frac{1}{2} \frac{v^2}{c^2} = \frac{1}{2} \frac{\alpha^2}{n^2} \quad -(24)$$

i) The Thomas half.

So in eq. (19):

$$H_0 = -\frac{1}{2} \frac{\alpha^2}{n^2} mc^2 + \frac{\underline{\sigma} \cdot \underline{p} \, U \, \underline{\sigma} \cdot \underline{p}}{4m^2c^2} \quad -(25)$$

This equation is often claimed to give the fine structure
of the H atom, but it is, nevertheless, an approximation.

The exact hamiltonian is:

4) $$H_0 = H - mc^2 = \frac{p^2}{m(1+\gamma)} + U$$

$$= \frac{p^2}{m\left(2 + \frac{\Delta\phi_T}{2\pi}\right)} + U \quad -(26)$$

$-(27)$

If $\frac{\Delta\phi_T}{4\pi} << 1$, then:

$$H_0 = \frac{p^2}{2m}\left(1 + \frac{\Delta\phi_T}{4\pi}\right)^{-1} + U \quad -(28)$$

$$\sim \frac{p^2}{2m}\left(1 - \frac{\Delta\phi_T}{4\pi}\right) + U$$

So the exact energy levels of the Dirac Hamiltonian are given by the Thomas precession.

Using the definition:

$$H = \gamma mc^2 + U \quad -(29)$$

then $$\gamma = \frac{H - U}{mc^2} \quad -(30)$$

and $$\frac{\Delta\phi_T}{4\pi} = \frac{1}{2}\left(\frac{H - U}{mc^2} - 1\right) \quad -(31)$$

From eqs. (28) and (31):

$$H_0 = \frac{p^2}{2m} + U - \underline{\sigma} \cdot \underline{p}\left(\frac{1}{2m}\left(\frac{H - U}{mc^2} - 1\right)\right)\underline{\sigma} \cdot \underline{p}$$

$$= \frac{p^2}{2m} + U + \underline{\sigma} \cdot \underline{p}\, \frac{U}{4m^2c^2}\, \underline{\sigma} \cdot \underline{p} - \frac{\underline{\sigma} \cdot \underline{p}}{2m}\left(\frac{H}{mc^2} - 1\right)\underline{\sigma} \cdot \underline{p} \quad -(32)$$

5) Comparing eqs. (19) and (32), there is an extra term in the exact result, the last term on the right hand side of eq. (32). Unless this extra term vanishes for some reason, it contributes to spectra. It is a fundamental result of the Thomas precession, together with the usually considered spin orbit term, the third term on the right hand side of eq. (32). The non relativistic limit is also a fundamental result of the Thomas precession, the first two terms on the right hand side of Eq. (32).

Using the usual rule:

$$p\,\phi = -i\hbar\,\underline{\nabla}\,\phi \quad -(33)$$

the new term due to the Dirac equation without this approximation is:

$$H_1\,\phi = -\frac{i\hbar}{2m}\,\underline{\sigma}\cdot\underline{\nabla}\left(\left(\frac{H}{mc^2}-1\right)\underline{\sigma}\cdot\underline{p}\,\phi\right) \quad -(34)$$

In general this is not zero. We have:

$$\underline{\nabla}\left(\frac{H}{mc^2}-1\right) = 0 \quad -(35)$$

because H is a constant, but

$$\underline{\nabla}\left(\left(\frac{H}{mc^2}-1\right)\underline{\sigma}\cdot\underline{p}\,\phi\right) \quad -(36)$$

$$= \left(\frac{H}{mc^2}-1\right)\underline{\nabla}\left(\underline{\sigma}\cdot\underline{p}\,\phi\right) + \left(\underline{\nabla}\left(\frac{H}{mc^2}-1\right)\right)\left(\underline{\sigma}\cdot\underline{p}\,\phi\right)$$

This is not zero in general and produces new spectral detail due to the Thomas precession.
