08(2): Relativistic Quantum Mechanics and the Thomas Half

The hamiltonian of the Sommerfeld and Dirac atom can be expressed as:

$$H = \gamma mc^2 + U \quad -(1)$$

and is defined by the Thomas half:

$$\frac{\Delta \phi_T}{2\pi} = \gamma - 1 \quad -(2)$$

so

$$H = \left(1 + \frac{\Delta \phi_T}{2\pi}\right) mc^2 + U \quad -(3)$$

It follows that:

$$H_0 = H - mc^2 = \frac{\Delta \phi_T}{2\pi} mc^2 + U \quad -(4)$$

$$= (\gamma - 1) mc^2 + U$$

$$:= T + U$$

in which the relativistic kinetic energy is:

$$T = (\gamma - 1) mc^2 = \frac{\Delta \phi_T}{2\pi} mc^2 \quad -(5)$$

The non relativistic kinetic energy is defined by:

$$T_0 = \left(\left(1 - \frac{v_N^2}{c^2}\right)^{-1/2} - 1\right) mc^2 \xrightarrow[v_N \ll c]{} \left(1 + \frac{1}{2}\frac{v_N^2}{c^2} - 1\right) mc^2 \quad -(6)$$

$$= \frac{1}{2} m v_N^2$$

so

$$T_0 = \frac{\Delta \phi_T}{2\pi} mc^2 = \frac{1}{2}\frac{v_N^2}{c^2} mc^2 \quad -(7)$$

) Both the relativistic and non-relativistic kinetic energies are defined by the Thomas half. So the energy levels of all atoms and molecules are defined by:

$$\langle H_0 \rangle = mc^2 \left\langle \frac{\Delta \phi_T}{2\pi} \right\rangle + \langle U \rangle \quad -(8)$$

so of all relativistic and non-relativistic levels. The transition from non-relativistic to relativistic atoms is defined by:

$$\frac{1}{2} \frac{v_N^2}{c^2} \longrightarrow \gamma - 1 \quad -(9)$$

In the Schroedinger atom:

$$\left\langle \frac{v_N^2}{c^2} \right\rangle = \frac{\alpha^2}{n^2} \quad -(10)$$

and

$$\langle \bar{U} \rangle = -\frac{\alpha^2}{n^2} mc^2 \quad -(11)$$

where α is the fine structure constant and n is the principal quantum number.

So from eq. (8):

$$\langle H_0 \rangle = mc^2 \left\langle \frac{\Delta \phi_T}{2\pi} \right\rangle - \frac{\alpha^2}{n^2} mc^2$$

$$= \frac{1}{2} \left\langle \frac{v_N^2}{c^2} \right\rangle mc^2 - \frac{\alpha^2}{n^2} mc^2$$

$$= \frac{1}{2} \frac{\alpha^2}{n^2} mc^2 - \frac{\alpha^2}{n^2} mc^2$$

$$= -\frac{1}{2} \frac{\alpha^2}{n^2} mc^2 \quad -(12)$$

The expectation value of the Thomas half is the

) Schroedinger H atom is:

$$\left\langle \frac{\Delta \phi_T}{2\pi} \right\rangle = \frac{1}{2}\frac{\alpha^2}{n^2} \quad -(13)$$

$$= \frac{1}{2}\left\langle \frac{v_N^2}{c^2} \right\rangle$$

so

$$\left\langle \frac{v_N^2}{c^2} \right\rangle = \frac{\alpha^2}{n^2} \quad -(14)$$

It follows that

$$\left\langle \frac{p^2}{2m} \right\rangle = mc^2\frac{\alpha^2}{n^2} \quad -(15)$$

The expectation value is defined by:

$$\left\langle \frac{p^2}{2m} \right\rangle = \int \psi^* \frac{\hat{p}^2}{2m} \psi \, d\tau \quad -(16)$$

$$= -\frac{\hbar^2}{2m}\int \psi^* \nabla^2 \psi \, d\tau$$

The wavefunctions are calculated from:

$$\langle H_0 \rangle := E = -\frac{\hbar^2}{2m}\nabla^2\psi - \frac{e^2}{4\pi\epsilon_0 r}\psi \quad -(17)$$

where

$$U = -\frac{e^2}{4\pi\epsilon_0 r} \quad -(18)$$

$$= -\frac{\alpha \hbar c}{r}$$

From the expression:

$$E^2 = \gamma^2 m^2 c^4 = c^2 p^2 + m^2 c^4 \quad -(19)$$

it follows that:

4)
$$H = \gamma m c^2 + U$$
$$= \left(c^2 p^2 + m^2 c^4\right)^{-1/2} + U \quad -(20)$$

So
$$(H-U)^2 = c^2 p^2 + m^2 c^4 \quad -(21)$$

The relativistic kinetic energy is:
$$T = (\gamma - 1) m c^2 = \frac{\Delta \phi_T}{2\pi} m c^2 \quad -(22)$$

The total relativistic energy is:
$$E = \gamma m c^2 = T + m c^2 \quad -(23)$$

So
$$E = H - U \quad -(24)$$

From eq. (21):
$$\left(E - m c^2\right)\left(E + m c^2\right) = p^2 c^2 \quad -(24)$$

and
$$E = \frac{p^2 c^2}{H - U + m c^2} + m c^2 \quad -(25)$$

Therefore
$$H = E + U = \frac{p^2 c^2}{H - U + m c^2} + m c^2 + U \quad -(26)$$

and
$$H_0 = H - m c^2 = \frac{p^2 c^2}{H - U + m c^2} + U \quad -(27)$$

Comparing eqs. (4) and (27): $\quad -(28)$
$$\boxed{\frac{\Delta \phi_T}{2\pi} = (\gamma - 1) = \frac{1}{m c^2} \frac{p^2 c^2}{H - U + m c^2}}$$

Eq. (28) is the route to the quantization of:

$$H_0 = (\gamma - 1) mc^2 + U \quad -(29)$$

$$= \frac{p^2 c^2}{H - U + mc^2} + U$$

So the Thomas half gives the energy levels of the Dirac equation, spin orbit coupling, and the electron g factor.

The usual route to quantization is to use the Dirac approximation of the previous note:

$$H_0 = \frac{p^2}{2m}\left(1 + \frac{U}{2mc^2}\right) + U \quad -(30)$$

So

$$H_0 \phi = -\frac{\hbar^2}{2m}\nabla^2\phi + \frac{1}{4m^2c^2}\underline{\sigma}\cdot\underline{p}\, U \underline{\sigma}\cdot\underline{p}\,\phi + U\phi \quad -(31)$$

$$= -\frac{\hbar^2}{2m}\nabla^2\phi + U\phi - \frac{i\hbar}{4m^2c^2}\underline{\sigma}\cdot\underline{\nabla}\left(U\underline{\sigma}\cdot\underline{p}\,\phi\right)$$

giving the spin orbit correction to the energy levels of the H atom.

The spin orbit fine structure of H is due entirely to the Thomas half defined in eq. (28). In the Schroedinger equation:

$$H_0\phi = -\frac{\hbar^2}{2m}\nabla^2\phi + U\phi \quad -(32)$$

$$\langle H_0 \rangle = -\frac{1}{2}\frac{\alpha^2}{n^2} mc^2 \quad -(33)$$

The fine structure is superimposed on eq. (33) and is worked accurately by Eq. (31). In addition, the Lamb shift appears in atomic H and is due to the effect of the

6) vacuum. In ECE2 unified field theory the Lamb shift is described by the following equation:

$$F = \frac{d}{dt}\left(\gamma m \dot{r}\right) = -\frac{\alpha \hbar c}{r^2} + \omega \phi \quad -(34)$$

where ω is the spin connection. The latter is found for the experimentally observed Lamb shift.

In some previous UFT papers eq. (29) has been expressed as:

$$H_0 = \frac{p^2 c^2}{T + mc^2} + U \quad -(35)$$

$$= \frac{p^2 c^2}{(\gamma + 1) mc^2} + U$$

$$= \frac{p^2}{m(\gamma + 1)} + U$$

which can now be written as:

$$H_0 = \frac{p^2}{m\left(\frac{\Delta \phi_T}{2\pi} + 2\right)} + U \quad -(36)$$

The classical result:

$$H_0 = \frac{p^2}{2m} + U \quad -(37)$$

is obtained when

$$\frac{\Delta \phi_T}{2\pi} \rightarrow 0 \quad -(38)$$