407(4): Development of the ECE2 Theory of the H Atom of Note 407(1)

The fundamental new insight is that:

$$\boxed{\frac{\langle v \rangle}{c} = \frac{\alpha}{n}} \quad -(1)$$

where $\langle v \rangle$ is the expectation value of the orbital velocity of the electron in the H atom, α is the fine structure constant and n the principal quantum number.

This is a major discovery that can be developed in many ways, and was derived using Thomas precession theory. The expectation value of the Thomas half is:

$$\frac{1}{2}\frac{\langle v^2 \rangle}{c^2} = \frac{1}{2}\frac{\alpha^2}{n^2} \quad -(2)$$

Eq. (1) is the limit of a relativistic theory:

$$\frac{\Delta\phi}{2\pi} \sim \frac{1}{2}\frac{v^2}{c^2} \longrightarrow \gamma - 1 \quad -(3)$$

$$= \left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1$$

Now note that the nonrelativistic kinetic energy is:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}\frac{v^2}{c^2}mc^2 \quad -(4)$$

It is the Thomas half, $\frac{1}{2}(v^2/c^2)$, multiplied by the rest energy mc^2. It follows that the relativistic kinetic energy is

$$T = (\gamma - 1)mc^2 \quad -(5)$$

$$= \left(\frac{\Delta\phi}{2\pi}\right)mc^2$$

which is the Thomas precession per radian multiplied by the rest energy mc^2.

Eq. (4) corresponds to:

$$H_0 = \frac{1}{2} m v^2 + U \quad -(6)$$

$$\mathcal{L}_0 = \frac{1}{2} m v^2 - U \quad -(7)$$

and Eq. (5) corresponds to:

$$H = \gamma m c^2 + U \quad -(8)$$

$$\mathcal{L} = -\frac{m c^2}{\gamma} - U \quad -(9)$$

Eqs. (8) and (9) describe the Sommerfeld atom, which is the relativistic Schroedinger atom.

In these equations:

$$U = -\frac{e^2}{4\pi\epsilon_0 r} = -\frac{\alpha \hbar c}{r} \quad -(10)$$

and

$$\langle U \rangle = -m\left(\frac{\alpha c}{n}\right)^2 = -m\frac{\langle v^2 \rangle}{c^2} \quad -(11)$$

So

$$\frac{1}{2} m \langle v^2 \rangle = \frac{1}{2} m \left(\frac{\alpha c}{n}\right)^2 \quad -(12)$$

and the energy levels of the Schroedinger H atom are

$$E_n = \frac{1}{2} m \langle v^2 \rangle + \langle U \rangle$$

$$= -\frac{1}{2} m \langle v^2 \rangle$$

$$= -\frac{1}{2} m \left(\frac{\alpha c}{n}\right)^2 = -\frac{1}{2}\frac{\alpha^2}{n^2} m c^2 \quad -(13)$$

$$= -\frac{1}{2}\left(\frac{\langle v^2 \rangle}{c^2}\right) m c^2$$

The energy levels of the Schroedinger H atom are the negative of the expectation values of the electronic kinetic energy in each orbital.

It follows immediately that the energy levels of the Sommerfeld atom are given by the replacement:

$$\frac{1}{2} m \langle v^2 \rangle \rightarrow (\gamma - 1) mc^2 \quad -(14)$$

so they are given by:

$$En(\text{Sommerfeld}) = -(\gamma - 1) mc^2 \quad -(15)$$

where the expectation value of the Lorentz factor is:

$$\langle \gamma \rangle = \left(1 - \frac{\langle v^2 \rangle}{c^2}\right)^{-1/2} = \left(1 - \left(\frac{\alpha}{n}\right)^2\right)^{-1/2} \quad -(16)$$

so:

$$En(\text{Sommerfeld}) = -\left(\left(1 - \left(\frac{\alpha}{n}\right)^2\right)^{-1/2} - 1\right) mc^2 \quad -(17)$$

$$= -\frac{\Delta \phi_T}{2\pi} mc^2$$

The energy levels of the Sommerfeld atom are the Thomas precession per radian multiplied by the rest energy.

The spin connection of the Sommerfeld atom is obtained from the lagrangian:

$$\mathcal{L} = -\left(1 - \frac{\dot{r}^2}{c^2}\right)^{1/2} mc^2 + \frac{\alpha \hbar c}{r} \quad -(18)$$

and the Euler Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(19)$$

From eqs. (18) and (19):

$$F = \frac{d}{dt}(\gamma m \dot{r}) = -\frac{\alpha \hbar c}{r^2} \quad -(20)$$

) as in UFT 238 and other papers. Therefore:

$$F = \gamma m \ddot{r} + \gamma m \dot{r} \frac{d\gamma}{dt} = - \frac{\alpha \hbar c}{r^2} \quad -(21)$$

Use:

$$\frac{d\gamma}{dt} = \frac{d\gamma}{dv} \frac{dv}{dt} = \frac{d\gamma}{dv} \ddot{r} \quad -(22)$$

so

$$F = \gamma m \dot{v} + m \gamma^3 \frac{v^2}{c^2}$$

$$= \gamma m \dot{v} \left(1 + \left(\frac{\gamma v}{c} \right)^2 \right)$$

$$= \gamma^3 m \dot{v} \left(\frac{1}{\gamma^2} + \frac{v^2}{c^2} \right) \quad -(23)$$

$$= \gamma^3 m \dot{v},$$

so the force equation is:

$$F = \gamma^3 m \ddot{r} = - \frac{\alpha \hbar c}{r^2} \quad -(24)$$

In ECE2 theory:

$$F = -\nabla \phi + m \omega \phi = \gamma^3 m \ddot{r} \quad -(25)$$

and the spin connection can be evaluated as in previous work. We now have the new insight that the relativistic force is

$$\langle F \rangle = - \alpha \hbar c \left\langle \frac{1}{r^2} \right\rangle \quad -(26)$$

and

$$\langle F \rangle = \langle \gamma^3 \rangle m \ddot{r} = \left(1 - \left(\frac{\alpha}{n} \right)^2 \right)^{-3/2} m \ddot{r} \quad -(27)$$

Therefore $\langle \ddot{r} \rangle$ can be calculated:

5)

$$\langle \ddot{r} \rangle = -\frac{\alpha \hbar c}{m} \left(1 - \left(\frac{\alpha}{n} \right)^2 \right)^{3/2} \quad -(28)$$

given the expectation value $\left\langle \frac{1}{r^2} \right\rangle$ for the H atom.

The energy levels of the H atom can also be worked out with the Dirac equation, with spin orbit coupling included.