03(a): Analytical Solution in the Near Circular Approximation

· The correction factor x is:

$$x = \frac{4}{3}\frac{\langle \delta \underline{r} \cdot \delta \underline{r} \rangle u^3}{-\alpha u^2 + 2u + \frac{1}{a}} \log_e u \quad -(1)$$

and produces:

$$\Delta \phi = \frac{2}{3}\alpha^{1/2}\langle \delta \underline{r} \cdot \delta \underline{r} \rangle \int \left(\frac{u^2}{-\alpha u^2 + 2u + \frac{1}{a}}\right)^{3/2} \log_e u \, du \quad -(2)$$

this produces a small correction to the ellipse:

$$r = \frac{\alpha}{1 + \epsilon \cos \phi} \quad -(3)$$

i.e

$$1 + \epsilon \cos \phi = \alpha u \quad -(4)$$

so

$$\phi = \cos^{-1}\left(\frac{1}{\epsilon}(\alpha u - 1)\right) \quad -(5)$$

so the complete function is

$$\phi_1 = \phi + \Delta \phi \quad -(6)$$

From the original method in the near circular approximation:

$$\Delta \phi = \frac{4}{3}\langle \delta \underline{r} \cdot \delta \underline{r} \rangle u^2 \quad -(7)$$

and at the perihelion:

$$u^2 = \frac{1}{a^2(1-\epsilon)^2} \quad -(8)$$

Eq. (8) is a special case of eq. (2) when:

2)

$$\alpha^{1/2} \int \left(\frac{u^2}{-\alpha u^2 + 2u + \frac{1}{a}} \right)^{3/2} \log_e u \, du = \frac{2}{a^2 (1-\epsilon)^2} \quad -(9)$$

where

$$a(1-\epsilon) = \frac{\alpha}{1+\epsilon} \quad -(10)$$

In the near circular approximation:

$$u^{-1} \sim \alpha \quad -(11)$$

So eq. (9) reduces to:

$$\alpha^{1/2} \int \left(\frac{u^2}{u + 1/a} \right)^{3/2} \log_e u \, du = \frac{2}{a^2 (1-\epsilon)^2} \quad -(10)$$

$$= \frac{2(1+\epsilon)^2}{\alpha^2} \quad -(11)$$

So

$$\int \left(\frac{u^2}{u + 1/a} \right)^{3/2} \log_e u \, du = \frac{2(1+\epsilon)^2}{\alpha^{5/2}}$$

Under condition (11) the results of the original method are obtained.

Wolfram integrator gives an analytical result for the integral (11). Denoting:

$$b = 1/a \quad -(12)$$

then:

$$f(u) := \int \left(\frac{u^2}{u+b} \right)^{3/2} \log_e u \, du$$

$$= \frac{1}{75 u^2} \left(\frac{u^2}{u+b} \right)^{1/2} \Big[480 b^{5/2} (u+b)^{1/2} \tanh^{-1} \left(\left(\frac{u+b}{b} \right)^{1/2} \right.$$

$$-296b^3 - 268b^2u + 15\left(16b^3 + 8b^2u - 2bu^2 + u^3\right)\log_e u + 22bu^2 - 6u^3 \Big] + A \quad -(13)$$

where A is a constant of integration.

At the perihelion:

$$u = \frac{1+\epsilon}{\alpha} = \frac{1}{a(1-\epsilon)} \quad -(14)$$

So A is adjusted to give agreement between the two sides of eq. (11) at the perihelion. The orbit in the near circular approximation is:

$$\phi = \cos^{-1}\left(\frac{1}{\epsilon}\left(\alpha u - 1\right)\right) + \Delta\phi \quad -(15)$$

where

$$\Delta\phi = \frac{2}{3}\alpha^{1/2}\langle \delta\underline{r}\cdot\delta\underline{r}\rangle f(u) \quad -(16)$$

So the orbit (15) can be plotted and compared with the ellipse:

$$\phi_0 = \cos^{-1}\left(\frac{1}{\epsilon}\left(\alpha u - 1\right)\right) \quad -(17)$$