403(1): Analytical Approximation to the ECE2 Equation of Orbits

Consider the ECE2 equation of orbits:

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{1}{\alpha}\left(1 - x\right)^{-1/2} \quad -(1)$$

in plane polar coordinates (r, ϕ). Here:

$$x = \frac{L^2}{m^2 c^2}\left(\frac{1}{r^2} + \left(\frac{d}{d\phi}\left(\frac{1}{r}\right)\right)^2\right) \quad -(2)$$

in which

$$L = \gamma m r^2 \dot{\phi} \quad -(3)$$

is the relativistic angular momentum, a constant of motion:

$$\frac{dL}{dt} = 0. \quad -(4)$$

The mass m orbits a mass M according to the inverse square force law:

$$F = -\frac{mMG}{r^2} \quad -(5)$$

This note shows that eq. (1) produces orbital precession.

In the non relativistic limit:

$$x \rightarrow 0 \quad -(6)$$

so eq. (1) reduces to the Binet equation of a static ellipse:

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{1}{\alpha} \quad -(7)$$

in which the half right latitude is:

$$\alpha = \frac{L^2}{m^2 M G} \quad -(8)$$

The half right latitude of the ellipse is defined by the function

$$r = \frac{\alpha}{1 + \epsilon\cos\phi} \quad -(9)$$

2) when

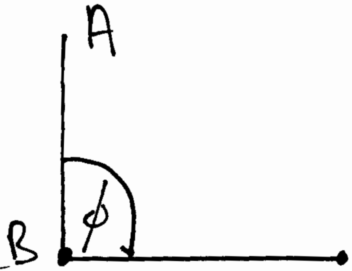
$$\phi = \frac{\pi}{2} \quad - (10)$$

so

$$r = \alpha \quad - (11)$$

as shown in Fig (1)

Fig. (1)



in which A is a point on the ellipse and B is a focus of the ellipse, at which the mass M is situated. Since α is a constant of the orbit:

$$\frac{d\alpha}{dt} = 0 \quad - (12)$$

so from eq. (11)

$$\frac{dr}{dt} = \dot{r} = 0 \quad - (13)$$

and

$$\frac{du}{d\phi} = -\frac{m}{L}\frac{dr}{dt} = 0 \quad - (14)$$

It follows that

$$\frac{d^2u}{d\phi^2} = -\frac{m^2}{L^2} r^2 \ddot{r} = 0 \quad - (15)$$

From eqs. (7) and (15):

$$r = \alpha \quad - (16)$$

which is eq. (11), Q.E.D.

For small precessions, eq. (2) is well approximated by eq. (7), and at the point defined by eq. (11),

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) = \frac{d}{d\phi}\left(\frac{1}{r}\right) = 0 \quad -(10)$$

eq. (1). So eq. (1) becomes:

$$\frac{1}{r} = \frac{1}{\alpha}\left(1 - \frac{L^2}{m^2c^2r^2}\right)^{-1/2} \quad -(11)$$

i.e.

$$\frac{r^2}{\alpha^2} = 1 - \frac{L^2}{m^2c^2r^2} \quad -(12)$$

so the ratio r/α is less than one and the point of the half right latitude has moved clockwise, i.e. <u>the ellipse has undergone a precession.</u>

Solving eq. (12):

$$r^2 = \frac{\alpha}{2}\left(\alpha \pm \left(\alpha^2 - \frac{4L^2}{m^2c^2}\right)^{1/2}\right) \quad -(13)$$

If:

$$\frac{4L^2}{m^2c^2} \longrightarrow 0 \quad -(14)$$

then the positive sign is indicated in eq. (13) because:

$$r^2 \longrightarrow \frac{\alpha}{2}(\alpha + \alpha) = \alpha^2 \quad -(15)$$

i.e.

$$r = \alpha \quad -(16)$$

Denote:

$$r_1 = \left(\frac{\alpha}{2}\left(\alpha + \left(\alpha^2 - \frac{4L^2}{m^2c^2}\right)^{1/2}\right)\right)^{1/2} \quad -(17)$$

and it is clear that:

$$r_1 < r \quad -(18)$$

The overall result is illustrated in Fig. (2):

4)

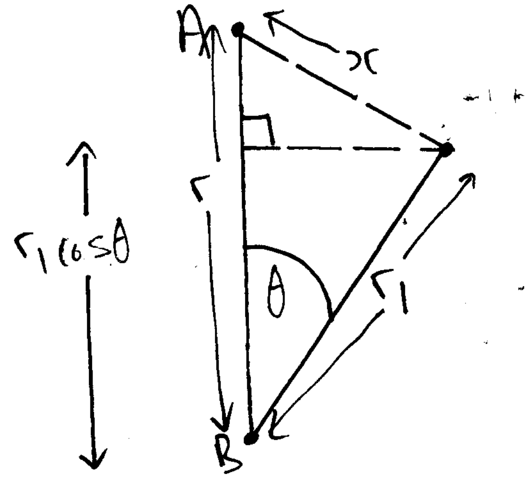


Fig (2)

Here θ is the angle of precession.

From the geometry of triangles:

$$x^2 = r^2 + r_1^2 - 2rr_1\cos\theta \quad — (19)$$

and applies to the larger triangle in Fig. (2). The smaller, right angled, triangle is defined in Fig (3):

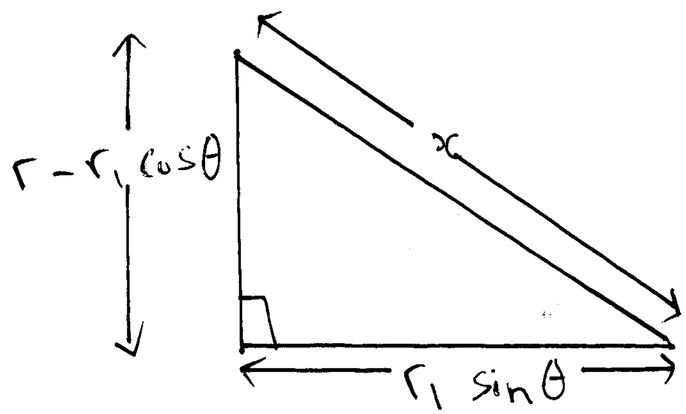


Fig (3)

So

$$x^2 = (r - r_1\cos\theta)^2 + r_1^2\sin^2\theta \quad — (20)$$

Eliminating x^2 in eqs. (19) and (20) gives θ in terms of r and r_1, so the precession angle can be found. So:

$$r^2 + r_1^2 - 2rr_1\cos\theta = (r - r_1\cos\theta)^2 + r_1^2\sin^2\theta \quad — (21)$$

$$= r^2 - 2rr_1\cos\theta + r_1^2(\sin^2\theta + \cos^2\theta)$$

5)

$$= r^2 - 2rr_1 \cos\theta + r_1^2$$

So the left and right hand sides are equal and eqs (19) and (20) are correct. Eq. (19) is the triangle formula for two sides and an included angle, and eq. (20) is the Pythagoras Theorem.

From eq. (19):

$$r_1^2 - 2rr_1\cos\theta + r^2 - x^2 = 0 \quad -(22)$$

so

$$r_1 = \frac{1}{2}\left(2r\cos\theta \pm \left(4r^2\cos^2\theta - 4(r^2 - x^2)\right)^{1/2}\right)$$

$$= r\cos\theta \pm \left(r^2\cos^2\theta - (r^2 - x^2)\right)^{1/2} \quad -(23)$$

For small precessions:

$$x \rightarrow 0, \quad \cos\theta \rightarrow 1 \quad -(24)$$

so

$$\boxed{\cos\theta = \frac{r_1}{r}} \quad -(25)$$

The angle of precession is defined by:

$$\cos\theta = \frac{1}{r}\left(\frac{\alpha}{2}\left(\alpha + \left(\alpha^2 - \frac{4L^2}{m^2c^2}\right)^{1/2}\right)\right)^{1/2} \quad -(26)$$

$$\xrightarrow[L \to 0]{} \frac{\alpha}{r}$$

The half right latitude of the static ellipse is defined by

$$\alpha = r \quad -(27)$$

so in the static ellipse:

$$\cos\theta = 1, \quad \theta = 0 \quad -(28)$$

) and there is no precession, Q.E.D., a self consistent result.

However, from eq. (26), using eq. (27):

$$\boxed{\cos\theta = \frac{1}{\alpha}\left(\frac{\alpha}{2}\left(\alpha + \left(\alpha^2 - \frac{4L^2}{m^2c^2}\right)^{1/2}\right)\right)^{1/2}} \quad -(29)$$

$$\longrightarrow 1$$

as

$$\frac{4L^2}{m^2c^2} \rightarrow 0 \quad -(30)$$

The exact precession angle is found by varying the relativistic angular momentum:

$$L = \gamma m r^2 \frac{d\phi}{dt} \quad -(31)$$

with

$$\frac{dL}{dt} = 0 \quad -(32)$$