3(7): Protocol Solution of the ECE2 Force Equation.

. Use of Maxima shows that the general solution of the ECE2 force equation:

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{1}{\alpha}\left(1+r\omega_r\right) \quad -(1)$$

$$\phi=-\int\left(\frac{\alpha}{2\omega_r\log_e u-\alpha u^2+2u-2C_1}\right)^{1/2}du \quad -(2)$$

here

$$u=\frac{1}{r} \quad -(3)$$

and

$$\alpha=\frac{L^2}{m^2MG} \quad -(4)$$

In the limit:

$$\omega_r\rightarrow 0 \quad -(5)$$

eq. (2) reduces to:

$$\phi=-\int\left(\frac{\alpha}{-\alpha u^2+2u-2C_1}\right)^{1/2}du \quad -(6)$$

Now compare eq. (6) with the Newtonian equation:

$$\phi(r)=\int\frac{L}{r^2}\frac{dr}{\left(2m\left(H+\frac{mMG}{r}-\frac{L^2}{2mr^2}\right)\right)^{1/2}} \quad -(7)$$

which is obtained from the Newtonian hamiltonian H in plane polar coordinates:

$$H=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\phi}^2\right)-\frac{mMG}{r} \quad -(8)$$

$$=\frac{1}{2}m\left(\dot{r}^2+\frac{L^2}{m r^2}\right)-\frac{mMG}{r}$$

from which:

$$\frac{dr}{dt} = \left(\frac{2}{m} \left(H + \frac{mMG}{r} \right) - \frac{L^2}{m^2 r^2} \right)^{1/2} \quad -(7)$$

$$= \frac{dr}{d\phi} \frac{d\phi}{dt} = \frac{L}{mr^2} \frac{dr}{d\phi}$$

$$-(8)$$

so
$$\phi(r) = \int \frac{L}{r^2} \frac{dr}{\left(2m \left(H + \frac{mMG}{r} - \frac{L^2}{2mr^2} \right) \right)^{1/2}}$$

which is eq. (7), Q.E.D.

Now use:
$$u = \frac{1}{r} \quad -(9)$$

to transform eq. (8) using:
$$\frac{du}{dr} = -\frac{1}{r^2} \quad -(10)$$

$$-(11)$$

so
$$\phi(u) = -L \int \frac{du}{\left(2m \left(H + \frac{mMG}{r} - \frac{L^2}{2m} u^2 \right) \right)^{1/2}}$$

which is eq. (6) with
$$C_1 = \frac{H}{mMG} \quad -(12)$$

Q.E.D. The solution of eqs. (11) and (6) is the conic section

$$\frac{\alpha}{r} = 1 + \epsilon \cos \phi \quad -(13)$$

where the half right latitude is:

$$\alpha = \frac{L^2}{m^2 M G} \quad -(14)$$

and

$$\epsilon = \left(1 + \frac{2HL^2}{m^3 M^2 G^2}\right)^{1/2} \quad -(15)$$

$$= \left(1 + \frac{C_1 L^2}{m^2 M G}\right)^{1/2}$$

$$= \left(1 + C_1 \alpha\right)^{1/2}$$

is the eccentricity.

So the solution of eq. (2) is a small perturbation of the conic section in the limit:

$$\omega_r \rightarrow 0 \quad -(16)$$

in the solar section. It is known from the previous note that this small perturbation is a precession.

Now use:

$$2\omega_r \log_e u - \alpha u^2 + 2u - 2C_1 \quad -(17)$$

$$= \left(-\alpha u^2 + 2u - 2C_1\right)\left(1 + \frac{2\omega_r \log_e u}{-\alpha u^2 + 2u - 2C_1}\right)$$

Then:

$$\phi = -\int \left(\frac{\alpha}{\left(-\alpha u^2 + 2u - 2C_1\right)\left(1 + x\right)}\right)^{1/2} du \quad -(18)$$

where:

$$x = \frac{2\omega_r \log_e u}{-\alpha u^2 + 2u - 2C_1} \quad -(19)$$

$$\ll 1$$

So:

$$\phi = -\int \left(\frac{\alpha}{-\alpha u^2 + 2u - 2C_1}\right)^{1/2} \left(1 - \frac{x}{2}\right)^{1/2} du \quad -(20)$$

$$\sim -\int \left(\frac{\alpha}{-\alpha u^2 + 2u - 2C_1}\right)^{1/2} \left(1 - \frac{x}{4}\right) du$$

and the orbit is an ellipse plus a small correction given by

$$\Delta\phi = \int \frac{x}{4} \left(\frac{\alpha}{-\alpha u^2 + 2u - 2C_1}\right)^{1/2} du \quad -(21)$$

$$= \int \frac{x}{4} \frac{\alpha^{1/2}}{(-\alpha u^2 + 2u - 2C_1)} du$$

$$= \frac{1}{2} \omega_r \alpha^{1/2} \int \frac{\log_e u \; du}{(-\alpha u^2 + 2u - 2C_1)}$$

The remaining task is to evaluate this integral, and the orbit is:

$$\phi = \phi_0 + \Delta\phi \quad -(22)$$

where

$$r = \frac{\alpha}{1 + \epsilon \cos\phi_0} \quad -(23)$$