403(10): Development of Note 403(6)

In the near circular approximation, the apsidal method gives an orbital precession of:

$$\Delta \phi = \frac{1}{3} \frac{\langle \delta \underline{r} \cdot \delta \underline{r} \rangle}{r^2} - \frac{r^2}{2} \frac{d\omega}{dr}, \quad -(1)$$

which is eq. (41) of Note 403(6). In eq. (1):

$$\omega = \frac{2}{3} \frac{\langle \delta \underline{r} \cdot \delta \underline{r} \rangle}{r^3} \quad -(2)$$

So

$$\frac{d\omega}{dr} = -2 \frac{\langle \delta \underline{r} \cdot \delta \underline{r} \rangle}{r^4} + \frac{2}{3} \frac{1}{r^3} \frac{\partial}{\partial r} \langle \delta \underline{r} \cdot \delta \underline{r} \rangle \quad -(3)$$

So:

$$\boxed{\Delta \phi = \frac{\langle \delta \underline{r} \cdot \delta \underline{r} \rangle}{r^2} - \frac{1}{3r} \frac{\partial}{\partial r} \langle \delta \underline{r} \cdot \delta \underline{r} \rangle} \quad -(4)$$

At the perihelion:

$$r = r_{min} = a(1-\epsilon) = \frac{\alpha}{1+\epsilon} \quad -(5)$$

where a is the semi major axis, α is the half right latitude and ϵ the eccentricity. In the limit of an exactly circular orbit:

$$\epsilon \rightarrow 0 \quad -(6)$$

so

$$r_{min} = a = \alpha \quad -(7)$$

If it is assumed that for an exactly circular orbit:

$$\Delta \phi = 0 \quad -(8)$$

2) then eq. (4) becomes:

$$\frac{1}{3a}\frac{\partial}{\partial r}\langle \delta \underline{r} \cdot \delta \underline{r} \rangle = \frac{\langle \delta \underline{r} \cdot \delta \underline{r} \rangle}{a^2} \quad -(9)$$

so

$$\frac{\partial}{\partial r}\langle \delta \underline{r} \cdot \delta \underline{r} \rangle = \frac{3}{a}\langle \delta \underline{r} \cdot \delta \underline{r} \rangle \quad -(10)$$

and

$$\Delta \phi = 0 \quad -(11)$$

Eq. (11) follows from eqs. (4) and (10).

If it is assumed that eq. (10) is approximately true in a near circular orbit, then eq. (4) becomes:

$$\Delta \phi \sim \frac{\langle \delta \underline{r} \cdot \delta \underline{r} \rangle}{r^2} - \frac{1}{ar}\langle \delta \underline{r} \cdot \delta \underline{r} \rangle .$$

$$= \frac{\langle \delta \underline{r} \cdot \delta \underline{r} \rangle}{r}\left(\frac{1}{r} - \frac{1}{a}\right) \quad -(12)$$

The exact result is eq. (4), and can be compared with eq. (2) of Note 403(9):

$$\Delta \phi = \frac{2}{3}\alpha^{1/2}\langle \delta \underline{r} \cdot \delta \underline{r} \rangle \int \left(\frac{u^2}{-\alpha u^2 + 2u + \frac{1}{a}}\right)^{3/2} \log_e du \quad -(13)$$

which has been integrated numerically by co author Horst Eckardt. Comparing eqs. (4) and (13), the quantity $\frac{\partial}{\partial r}\langle \delta \underline{r} \cdot \delta \underline{r} \rangle$ can be found from:

3)
$$\frac{\langle \delta \underline{r} \cdot \delta \underline{r} \rangle}{r^2} - \frac{1}{3r}\frac{\partial}{\partial r}\langle \delta \underline{r} \cdot \delta \underline{r} \rangle$$
$$= \frac{2}{3}\alpha^{1/2}\langle \delta \underline{r} \cdot \delta \underline{r} \rangle \int \left(\frac{u^2}{-\alpha u^2 + 2u + \frac{1}{a}} \right)^{3/2} \log_e du$$
$$- (14)$$

Eq. (14) is a comparison of precessions using analytical approximations in the near circular limit.

The orbit is:
$$\phi = -\int \left(\frac{\alpha}{2\omega_r \log_e u - \alpha u^2 + 2u + \frac{1}{a}} \right)^{1/2} du.$$
$$- (15)$$

In the limit:
$$\omega_r \rightarrow 0 \quad - (16)$$

the orbit (15) is the conic section:
$$r = \frac{\alpha}{1 + \epsilon \cos \phi} \quad - (17)$$

In eq. (15):
$$\omega_r = \frac{2}{3}\frac{\langle \delta \underline{r} \cdot \delta \underline{r} \rangle}{r^3} \quad - (18)$$

so the well known conic section is perturbed by the isotropically averaged vacuum fluctuation $\langle \delta \underline{r} \cdot \delta \underline{r} \rangle$. The vacuum perturbation produces precession given by eq. (14) in the near circular limit.