403(5): Approximate Solution for $\epsilon\cos\phi < 1$

Consider the differential equation:

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r_2}\right)+\frac{1}{r_2}=\frac{\omega_r}{1+\epsilon\cos\phi} \quad -(1)$$

Then the general solution of the ECE2 force equation:

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{1}{\alpha}\left(1+r\omega_r\right) \quad -(2)$$

is

$$\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2} \quad -(3)$$

$$=\frac{1}{\alpha}\left(1+\epsilon\cos\phi\right)+\frac{1}{r_2}$$

in the method of successive approximations.

Eq. (1) does not appear to have a solution from the Wolfram integrator, but eqs. (1) and (2) can be integrated numerically by Maxima. Eq. (1) can be solved approximately if

$$\epsilon\cos\phi < 1 \quad -(4)$$

in which case:

$$\frac{\omega_r}{1+\epsilon\cos\phi}\sim\omega_r\left(1-\epsilon\cos\phi\right) \quad -(5)$$

and from the Wolfram solver:

$$\frac{1}{r_2}=-\frac{1}{2}\omega_r\epsilon\phi\sin\phi+C_2\sin\phi+C_1\cos\phi \quad -(6)$$

$$=\left(C_2-\frac{1}{2}\omega_r\epsilon\phi\right)\sin\phi+C_1\cos\phi$$

where C_1 and C_2 are constants. So the solution of eq. (2) is:

$$\boxed{\frac{1}{r}=\frac{1}{\alpha}\left(1+\epsilon\cos\phi\right)+\left(C_2-\frac{1}{2}\omega_r\epsilon\phi\right)\sin\phi+C_1\cos\phi} \quad -(7)$$

More accurately, eq. (5) is:

$$\frac{\omega_r}{1+\epsilon\cos\phi} \sim \omega_r\left(1-\epsilon\cos\phi+\epsilon^2\cos^2\phi-\epsilon^3\cos^3\phi+\ldots\right) \quad -(8)$$

If $$\frac{\omega_r}{1+\epsilon\cos\phi} \sim \omega_r\left(1-\epsilon\cos\phi+\epsilon^2\cos^2\phi\right)$$

then

$$\frac{1}{r_2} = -\frac{1}{6}\omega_r\epsilon^2\cos(2\phi)+\frac{\omega_r\epsilon^2}{2}-\frac{1}{2}\omega_r\epsilon\phi\sin\phi + \omega_r + C_2\sin\phi + C_1\cos\phi$$

$$= \omega_r + \frac{\omega_r\epsilon^2}{2}\left(1-\frac{1}{3}\cos(2\phi)\right) + \left(C_2-\frac{1}{2}\omega_r\epsilon\phi\right)\sin\phi + C_1\cos\phi \quad -(9).$$

So

$$\frac{1}{r} = \frac{1}{\alpha}\left(1+\epsilon\cos\phi\right)+\omega_r+\frac{\omega_r\epsilon^2}{2}\left(1-\frac{1}{3}\cos(2\phi)\right) + \left(C_2-\frac{1}{2}\omega_r\epsilon\phi\right)\sin\phi + C_1\cos\phi \quad -(10)$$

This is already an adequate approximation because the eccentricity of the earth for example is

$$0.0034 < \epsilon < 0.058 \quad -(11)$$

and

$$-1 < \cos\phi < 1 \quad -(12)$$

Using the half right latitude in eq. (7)

$$\phi = \pi/2 \quad -(13)$$

then

$$\frac{1}{r} = \frac{1}{\alpha} + C_1 \quad -(14)$$

$$= \frac{1+\alpha C_1}{\alpha}$$

and the precession angle θ at the half right latitude can be calculated as follows.

If $r = \alpha$ and $R = \alpha / (1 + C_1 \alpha)$ then:

$$R < r \quad - (9)$$

and the precession angle is defined by Fig (1): as in Note 403(2).

By the triangle rule:

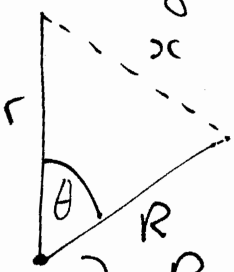

Fig (1)

$$x^2 = r^2 + R^2 - 2rR\cos\theta \quad - (10)$$

So

$$R = r\cos\theta \pm \left(r^2\cos^2\theta - (r^2 - x^2) \right)^{1/2} \quad - (11)$$

For small precessions:

$$x \rightarrow 0, \quad \cos\theta \rightarrow 1 \quad - (12)$$

so

$$\cos\theta \sim \frac{R}{r} = \frac{1}{1 + C_1\alpha} \quad - (13)$$

For

$$C_1\alpha << 1 : \quad - (14)$$

$$\cos\theta \sim 1 - C_1\alpha = 1 - \frac{\theta^2}{2} + \cdots \quad - (15)$$

so

$$\boxed{\theta = (2C_1\alpha)^{1/2}} \quad - (16)$$

For small precessions the half right latitude α is the half right latitude of the ellipse to an excellent approximation so the constant C_1 can be chosen that fit the astronomically observed θ.