403(3): Analytical Precessing Ellipse from the ECE2 Force Equation.

As in Note 403(2), the ECE2 force equation of orbits produces the ECE2 Binet equation:

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{1}{\alpha}\left(1+r\omega_r\right) \quad -(1)$$

in which ω_r is the radial component of the vector spin connection:

$$\underline{\omega}=\omega_r\,\underline{e}_r \quad -(2)$$

and α is the half right latitude of the ellipse.

Eq. (1) can be solved by successive approximation. In the first approximation:

$$\frac{1}{r}=\frac{1}{r_1}=\frac{1}{\alpha}\left(1+\epsilon\cos\phi\right) \quad -(3)$$

in which case:

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r_1}\right)+\frac{1}{r_1}=\frac{1}{\alpha} \quad -(4)$$

and

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{1}{\alpha}+\frac{\omega_r}{1+\epsilon\cos\phi} \quad -(5)$$

The right hand side of eq. (5) is the second term on the ellipse like function:

$$\frac{1}{r_3}:=\frac{\omega_r}{1+\epsilon\cos\phi} \quad -(6)$$

in which ω_r has the units of inverse metres. In the next level of approximation:

$$\frac{1}{r}:=\frac{1}{r_1}+\frac{1}{r_2} \quad -(7)$$

2) So:

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{d^2}{d\phi^2}\left(\frac{1}{r_1}\right)+\frac{1}{r_1}+\frac{d^2}{d\phi^2}\left(\frac{1}{r_2}\right)+\frac{1}{r_2}$$

$$=\frac{1}{\alpha}+\frac{\omega_r}{1+\epsilon\cos\phi} \quad -(8)$$

Therefore:

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r_2}\right)+\frac{1}{r_2}=\frac{\omega_r}{1+\epsilon\cos\phi}:=\frac{1}{r_3} \quad -(9)$$

This is a second order differential equation for $1/r_2$. Its solution is the sum of the complementary function and particular integral. The complementary function is the solution of:

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r_2}\right)+\frac{1}{r_2}=0 \quad -(10)$$

i.e.

$$\frac{1}{r_2}=C_1\cos\phi+C_2\sin\phi \quad -(11)$$

where C_1 and C_2 are constants.

By inspection, consider the solution:

$$y:=\frac{1}{r_2}=\frac{\omega_r}{1+|\epsilon|\cos\phi} \quad -(12)$$

of eq. (9). The eccentricity ϵ must be positive, so the modulus $|\epsilon|$ is used. It follows from eq. (12) that

$$\frac{d^2y}{d\phi^2}+y=y \quad -(13)$$

so

$$\frac{d^2y}{d\phi^2}=0 \quad -(14)$$

Note that if

$$y = \frac{\omega_0}{1+\epsilon\cos\phi} \quad (15)$$

then

$$\frac{dy}{d\phi} = \frac{\epsilon y^2}{\omega_r}\sin\phi \quad (16)$$

and

$$\frac{d^2 y}{d\phi^2} = \frac{\epsilon}{\omega_r}\left(y^2\cos\phi + 2y\frac{dy}{d\phi}\sin\phi\right)$$

$$= \frac{\epsilon}{\omega_r}\left(y^2\cos\phi + 2\frac{\epsilon}{\omega_r}y^3\sin^2\phi\right)$$

$$= 0 \quad (17)$$

So

$$\cos\phi + 2\frac{\epsilon}{\omega_r}y\sin^2\phi = 0 \quad (18)$$

and

$$y = \frac{1}{r_2} = -\frac{2\omega_r}{\epsilon}\frac{\cos\phi}{\sin^2\phi} = \frac{\omega_r}{1+\epsilon\cos\phi} \quad (19)$$

Eq (19) is an equation for ϵ:

$$\epsilon = -\frac{2\cos\phi}{\sin^2\phi}\left(1+\epsilon\cos\phi\right) \quad (20)$$

So

$$\epsilon\left(1+\frac{2\cos^2\phi}{\sin^2\phi}\right) = -\frac{2\cos\phi}{\sin^2\phi} \quad (21)$$

i.e.

$$\epsilon\left(1+\cos^2\phi\right) = -2\cos\phi \quad (22)$$

so

$$\epsilon = -\frac{2\cos\phi}{1+\cos^2\phi} \quad (23)$$

and

$$|\epsilon| = \frac{2\cos\phi}{1+\cos^2\phi} \quad (24)$$

Therefore:

$$y = \frac{1}{r_2} = \frac{\omega_0}{\dfrac{1+2\cos^2\phi}{1+\cos^2\phi}}$$

$$= \omega_0 \left(\frac{1+\cos^2\phi}{(1+\cos\phi)^2} \right) \quad -(25)$$

The general solution of eq. (1) is therefore:

$$\boxed{\frac{1}{r} = \frac{1}{\alpha}\left(1+\epsilon\cos\phi\right) + \omega_0\left(\frac{1+\cos^2\phi}{(1+\cos\phi)^2}\right) + C_1\cos\phi + C_2\sin\phi} \quad -(26)$$

which is the sum of the particular integral and the complementary function.

At the half right latitude:

$$\phi = \frac{\pi}{2}, \quad \cos\phi = 0 \quad -(27)$$

and assuming for simplicity that:

$$C_1 = C_2 = 0 \quad -(28)$$

it follows that:

$$\frac{1}{r} = \frac{1}{\alpha} + \omega_r \quad -(29)$$

$$= \frac{1+\alpha\omega_r}{\alpha}$$

$$:= \frac{1}{R}$$

so $$R < r \quad - (\overset{30}{\cancel{14}})$$

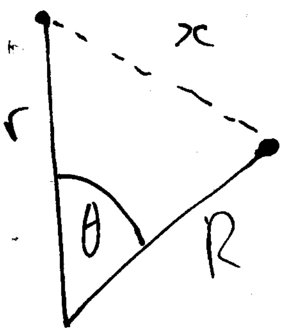

Fig (1)

and there is clockwise precession of the ball [illegible], as shown in Fig (1).

If ω_r is negative there is anticlockwise precession. Therefore:

1) $$R = \frac{\alpha}{1+\alpha\omega_r} \quad - (\overset{31}{\cancel{15}})$$

produces clockwise precession, and

2) $$R = \frac{\alpha}{1-\alpha\omega_r} \quad - (\overset{32}{\cancel{16}})$$

produces anticlockwise precession, as found in UFT401.

From the triangle rule in Fig (1):

$$x^2 = r^2 + R^2 - 2 r r_1 \cos\theta \quad - (\overset{33}{\cancel{17}})$$

so $$R = r\cos\theta \pm \left(r^2\cos^2\theta - (r^2 - x^2)\right)^{1/2} \quad - (\overset{34}{\cancel{18}})$$

For small θ: $$\cos^2\theta \to 1 \, ; \; x \to 0 \quad - (\overset{35}{\cancel{19}})$$

so $$\cos\theta \sim \frac{r_1}{R} \quad - (\overset{36}{\cancel{20}})$$

At the point $$r = \alpha \quad - (\overset{37}{\cancel{21}})$$

it follows that $$\boxed{\cos\theta = \frac{1}{1+\alpha\omega_r}} \quad - (\overset{38}{\cancel{22}})$$

For $$\alpha\omega_r \ll 1 \quad - (\overset{39}{\cancel{23}})$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \dots = 1 - \alpha\omega_r \quad - (\overset{40}{\cancel{24}})$$

which is eq. (33) of Note 403(2), Q.E.D.