403(8): Final Version of Note 403(7)

Consider the solution of the ECE2 force equation:

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{1}{\alpha}\left(1 + r\omega_r\right) \quad -(1)$$

given by Maxima computer algebra:

$$\phi = - \int \left(\frac{\alpha}{2\omega_r \log_e u - \alpha u^2 + 2u - 2C_1} \right)^{1/2} du \quad -(2)$$

where C_1 is a constant to be determined. Here:

$$u = \frac{1}{r}, \quad \alpha = \frac{L^2}{m^2 M G} \quad -(3)$$

In the limit:

$$\omega_r \rightarrow 0 \quad -(4)$$

Eq. (2) reduces to:

$$\phi = - \int \left(\frac{\alpha}{-\alpha u^2 + 2u - 2C_1} \right)^{1/2} du \quad -(5)$$

which is the equation of a conic section, notably an ellipse, provided that:

$$\phi(r) = \int \frac{L}{r^2} \frac{dr}{\left(2m\left(H + \frac{mMG}{r} - \frac{L^2}{2mr^2}\right)\right)^{1/2}} \quad -(6)$$

in the notation of Note 403(7).

Comparing eqs. (5) and (6):

$$\frac{L^2}{2m\left(H + mMGu - \frac{L^2}{2m}u^2\right)} = \frac{\alpha}{\left(-\alpha u^2 + 2u - 2C_1\right)} \quad -(7)$$

$$= \frac{L^2}{m^2 M G}\left(\frac{1}{\frac{2mH}{m^2 M G} + 2u - \frac{L^2 u^2}{m^2 M G}}\right)$$

$$= \frac{\alpha}{\frac{2H}{mMG} + 2u - \alpha u^2}$$

So the equations are the same if:

$$C_1 = -\frac{H}{mMG} \quad -(8)$$

Q.E.D.

The result (8) is also given by computer algebra.

The solution of eq. (6) is the conic section:

$$\frac{\alpha}{r} = 1 + \epsilon \cos\phi \quad -(9)$$

where the half right latitude is:

$$\alpha = \frac{L^2}{m^2 MG} \quad -(10)$$

and the eccentricity is:

$$\epsilon = \left(1 + \frac{2HL^2}{m^3 M^2 G^2}\right)^{1/2} \quad -(11)$$

The semi major axis is

$$a = \frac{\alpha}{1-\epsilon^2} = \frac{mMG}{2|H|} \quad -(12)$$

so from eqs (7) and (12):

$$\frac{1}{a} = \frac{2|H|}{m m G} \quad - (13)$$

So

$$\phi = -\int \left(\frac{\alpha}{-\alpha u^2 + 2u + \frac{1}{a}} \right)^{1/2} du \quad - (14)$$

is the equation of a static ellipse, as is well known. This is changed by the spin connection to give:

$$\phi = -\int \left(\frac{\alpha}{2\omega_r \log_e u - \alpha u^2 + 2u + \frac{1}{a}} \right)^{1/2} du \quad - (15)$$

Since ω_r is very small for small precessions, eq. (15) is a small perturbation of an ellipse. From note 405(6) it is known that this perturbation is the perihelion precession:

$$\Delta\phi = \frac{4}{3} \frac{\langle \delta \underline{r} \cdot \delta \underline{r} \rangle}{r^2} \quad - (16)$$

Now note that:

$$2\omega_r \log_e u - \alpha u^2 + 2u + \frac{1}{a}$$
$$= \left(-\alpha u^2 + 2u + \frac{1}{a} \right) \left(1 + x \right) \quad - (17)$$

Here:

$$x = \frac{2\omega_r \log_e u}{-\alpha u^2 + 2u + \frac{1}{a}} \quad - (18)$$

So:

$$\phi = -\int \left(\frac{\alpha}{\left(-\alpha u^2 + 2u + \frac{1}{a}\right)(1+x)} \right)^{1/2} du \quad - (19)$$

For small x:

$$(1+x)^{-1/2} \sim 1 - \frac{x}{2} \quad -(20)$$

so

$$\phi \sim -\int \left(\frac{\alpha}{-\alpha u^2 + 2u + \frac{1}{a}} \right)^{1/2} \left(1 - \frac{x}{2} \right) du \quad -(21)$$

$$= -\int \left(\frac{\alpha}{-\alpha u^2 + 2u + \frac{1}{a}} \right)^{1/2} + \frac{1}{2} \int x \left(\frac{\alpha}{-\alpha u^2 + 2u + \frac{1}{a}} \right)^{1/2} du$$

Therefore the correction to the ellipse is:

$$\Delta\phi = \frac{1}{2} \int x \left(\frac{\alpha}{-\alpha u^2 + 2u + \frac{1}{a}} \right)^{1/2} du$$

$$= \int \frac{\omega_r \alpha^{1/2} \log_e u \; du}{\left(-\alpha u^2 + 2u + \frac{1}{a} \right)^{3/2}} \quad -(22)$$

Now use:

$$\omega_r = \frac{2}{3} \frac{\langle \delta \underline{r} \cdot \delta \underline{r} \rangle}{r^3} = \frac{2}{3} \langle \delta \underline{r} \cdot \delta \underline{r} \rangle u^3 \quad -(23)$$

find that:

$$\Delta\phi = \frac{2}{3} \int \langle \delta \underline{r} \cdot \delta \underline{r} \rangle \left(\frac{u^2}{-\alpha u^2 + 2u + \frac{1}{a}} \right)^{3/2} \log_e u \; du \quad -(24)$$

If it is assumed that $\langle \delta \underline{r} \cdot \delta \underline{r} \rangle$ is approximately constant, then:

$$\Delta\phi \sim \frac{\alpha}{3}\langle \delta\underline{r}\cdot\delta\underline{r}\rangle \int \left(\frac{u^2}{-\alpha u^2+2u+\frac{1}{a}}\right)^{3/2} \log_e u \, du \quad -(25)$$

At the perihelion:

$$u_{min} = \frac{1+\epsilon}{\alpha} = \frac{1}{a(1-\epsilon)} \quad -(26)$$

and from eq. (16)

$$\Delta\phi(\text{perihelion}) = \frac{4}{3}\frac{\langle \delta\underline{r}\cdot\delta\underline{r}\rangle}{r_{min}^2}$$

$$= \frac{4}{3}\langle \delta\underline{r}\cdot\delta\underline{r}\rangle u_{min}^2$$

$$= \frac{4}{3}\left(\frac{1+\epsilon}{\alpha}\right)^2\langle \delta\underline{r}\cdot\delta\underline{r}\rangle \quad -(27)$$

Eq. (27) is a special case of eq. (25).

Maxima can be used to look for an analytical solution of eq. (25) or to integrate it numerically. It is seen that the precession at the perihelion is due to $\langle \delta\underline{r}\cdot\delta\underline{r}\rangle$.

Eq (25) has no solution in Wolfram but could be evaluated numerically or in an approximation.