398(4): Components of the Spin Connection Vector $\underline{\omega}$ for the Lamb Shift.

In ECE2 theory:

$$\underline{E} = -\underline{\nabla}\phi_0 + \underline{\omega}\phi_0 \quad -(1)$$

where $\underline{E}$ is the electric field strength and ϕ_0 the scalar potential. Here $\underline{\omega}$ is the spin connection vector:

$$\underline{\omega} = \omega_X \underline{i} + \omega_Y \underline{j} + \omega_Z \underline{k} \quad -(2)$$

In Eq. (1):

$$\underline{E}(vac) = \underline{\omega}\phi_0 \quad -(3)$$

is the vacuum electric field, and:

$$\underline{E}_0 = -\underline{\nabla}\phi_0 \quad -(4)$$

is the electric field strength in the hypothetical absence of the vacuum. So:

$$\langle \Delta \underline{E} \rangle = \underline{E} - \underline{E}_0 = \Delta \underline{E} = \underline{\omega}\phi_0 \quad -(5)$$

However, the tensor Taylor series gives:

$$\langle \Delta \underline{E} \rangle = \langle \Delta E_X \rangle \underline{i} + \langle \Delta E_Y \rangle \underline{j} + \langle \Delta E_Z \rangle \underline{k} \quad -(6)$$

in which:

$$\langle \Delta E_X \rangle = \langle \Delta E_X \rangle^{(2)} + \langle \Delta E_X \rangle^{(4)} + \langle \Delta E_X \rangle^{(6)} + \dots \quad -(7)$$

and similarly for the Y and Z components. The complete expressions are given by using:

$$f = E_X,\ E_Y,\ E_Z \quad -(8)$$

in Eqs. (2) to (4) of Note 398(1). For example at second order in the Taylor series:

$$\langle \Delta E_X \rangle^{(2)} = \frac{1}{6} \langle \delta \underline{r} \cdot \delta \underline{r} \rangle \nabla^2 E_X \quad -(9)$$

where

$$E_X = - \frac{e^2 X}{4\pi \epsilon_0 (X^2 + Y^2 + Z^2)^{3/2}} \quad -(10)$$

and similarly for the Y and Z components.

As in UFT 397:

$$\langle E_X \rangle^{(2)} = \langle E_Y \rangle^{(2)} = \langle E_Z \rangle^{(2)} = 0 \quad -(11)$$

so

$$\underline{\omega} \phi_0 = \langle \Delta E_X \rangle \underline{i} + \langle \Delta E_Y \rangle \underline{j} + \langle \Delta E_Z \rangle \underline{k} \quad -(12)$$

i.e.

$$-\frac{e^2}{4\pi \epsilon_0 r} \omega_X = \langle \Delta E_X \rangle^{(4)} + \langle \Delta E_X \rangle^{(6)} + \ldots \quad -(13)$$

$$-\frac{e^2}{4\pi \epsilon_0 r} \omega_Y = \langle \Delta E_Y \rangle^{(4)} + \langle \Delta E_Y \rangle^{(6)} + \ldots \quad -(14)$$

$$-\frac{e^2}{4\pi \epsilon_0 r} \omega_Z = \langle \Delta E_Z \rangle^{(4)} + \langle \Delta E_Z \rangle^{(4)} + \ldots \quad -(15)$$

where

$$r = (X^2 + Y^2 + Z^2)^{1/2} \quad -(16)$$

These are expressions for ω_X, ω_Y and ω_Z for the Coulombic interaction between the electron and proton in the H atom. The right hand sides of eqs. (13) to (15) are given by fluctuations after isotropic averaging by the expressions derived in the previous note:

$$\langle \delta \underline{r} \cdot \delta \underline{r} \rangle = \frac{2}{\pi} \alpha \lambda^2 \int_{\pi/a_0}^{mc/\hbar} \frac{d\kappa}{\kappa} = \frac{2}{\pi} \alpha \lambda^2 \log_e \frac{1}{\pi \alpha} \quad -(17)$$

$$\langle (\delta \underline{r} \cdot \delta \underline{r})^2 \rangle = \frac{4}{V} (\alpha \lambda^2)^2 \int_{\pi/a_0}^{mc/\hbar} \frac{d\kappa}{\kappa^4} \quad -(18)$$

$$\langle (\delta \underline{r} \cdot \delta \underline{r})^3 \rangle = \frac{8\pi}{V^2} (\alpha \lambda^2)^3 \int_{\pi/a_0}^{mc/\hbar} \frac{d\kappa}{\kappa^7} \quad -(19)$$

The Lamb shift is:

$$\langle \Delta \phi \rangle^{(2)} = \frac{1}{6} \langle \delta \underline{r} \cdot \delta \underline{r} \rangle \frac{e^2}{\epsilon_0} |\phi(0)|^2 \quad -(20)$$

$$= \frac{\hbar c}{6\pi a_0} \log_e \frac{1}{\pi \alpha}$$

and must be calculated with:

$$\nabla^2 \left(\frac{1}{r} \right) = -4\pi \delta_D \quad -(21)$$

where δ_D is the Dirac delta function. Note carefully that at the classical level of eqs. (13) to (15):

$$\nabla^2 \left(\frac{1}{r} \right) = 0 \quad -(22)$$

and

$$\nabla^2 \left(\frac{1}{r^2} \right) = 0 \quad -(23)$$

so eq. (11) follows.