398(2): Calculation of Higher Order Corrections to the Lamb Shift.

The basic idea of the Lamb shift is that:

$$m \frac{d^2 \delta \underline{r}}{dt^2} = -e\underline{E} \quad -(1)$$

where $-e$ is the charge on the electron. It can be seen immediately that the vacuum fluctuations generate an electric field strength. It is assumed that:

$$\delta r = \delta r(0) \exp(-i\omega t) -(2)$$

so

$$\frac{d^2 \delta r}{dt^2} = -\omega^2 \delta r \quad -(3)$$

and

$$m\omega^2 \delta r = eE \quad -(4)$$

so

$$\delta r = \frac{eE}{m\omega^2} = \frac{eE}{mc^2\kappa^2} \quad -(5)$$

It follows that

$$\langle \delta \underline{r} \cdot \delta \underline{r} \rangle = \sum_{\kappa} \left(\frac{e}{mc^2\kappa^2} \right)^2 \langle 0 | E^2 | 0 \rangle -(6)$$

where

$$\underline{E} = \underline{E}(0) \left(a_\kappa e^{-i(\omega t - \underline{\kappa} \cdot \underline{r})} + a_\kappa^+ e^{i(\omega t - \underline{\kappa} \cdot \underline{r})} \right) \quad -(7)$$

in standard mode theory.

It follows that:

$$\langle 0 | E^2 | 0 \rangle = \frac{\hbar c \kappa}{2\epsilon_0 V} -(8)$$

where V is the radiation volume.

So

$$\langle \delta \underline{r} \cdot \delta \underline{r} \rangle = \sum_{\kappa} \left(\frac{e}{mc^2 \kappa^2} \right)^2 \left(\frac{\hbar c \kappa}{2 \epsilon_0 V} \right) \quad -(9)$$

$$= \frac{2\pi}{V} \alpha \lambda^2 \sum_{\kappa} \frac{1}{\kappa^3}$$

Here α is the fine structure constant:

$$\alpha = \frac{e^2}{4\pi \hbar c \epsilon_0} \quad -(10)$$

and

$$\lambda = \frac{\hbar}{mc} \quad -(11)$$

Similarly:

$$\langle (\delta \underline{r} \cdot \delta \underline{r})^2 \rangle = \left(\frac{2\pi}{V} \alpha \lambda^2 \right)^2 \sum_{\kappa} \frac{1}{\kappa^6} \quad -(12)$$

and

$$\langle (\delta \underline{r} \cdot \delta \underline{r})^3 \rangle = \left(\frac{2\pi}{V} \alpha \lambda^2 \right)^3 \sum_{\kappa} \frac{1}{\kappa^9} \quad -(13)$$

In the theory of the Lamb shift, the summation in eq. (9) is replaced by an integral as follows:

$$\sum_{\kappa} \rightarrow \frac{2V}{(2\pi)^3} \int d^3\kappa = \frac{2V}{(2\pi)^3} \cdot 4\pi \int \kappa^2 d\kappa$$

$$= \frac{V}{\pi^2} \int \kappa^2 d\kappa \quad -(14)$$

From eqs. (9) and (14):

$$\langle \delta \underline{r} \cdot \delta \underline{r} \rangle = \frac{2}{\pi} \alpha \lambda^2 \int \frac{d\kappa}{\kappa} \quad -(15)$$

The lower bound of κ is:

$$\kappa_l = \frac{\pi}{a_0} = 5.936 \times 10^{10} \, m \quad -(16)$$

and the upper bound is:

$$\kappa_u = \frac{mc}{\hbar} = 6.570 \times 10^{10} \, m \quad -(17)$$

So:

$$\langle \delta \underline{r} \cdot \delta \underline{r} \rangle = \frac{2}{\pi} \alpha \lambda^2 \int_{\pi / a_0}^{mc/\hbar} \frac{d\kappa}{\kappa} \quad -(18)$$

$$= \frac{2}{\pi} \alpha \lambda^2 \left(\log_e \frac{mc}{\hbar} - \log_e \frac{\pi}{a_0} \right)$$

$$= \frac{2}{\pi} \alpha \lambda^2 \log_e \frac{mc a_0}{\pi \hbar}.$$

This result gives the Lamb shift with great accuracy.

From eqs. (5) and (8), higher order terms are given as follows:

$$\langle \delta \underline{r} \cdot \delta \underline{r} \, \delta \underline{r} \cdot \delta \underline{r} \rangle = \sum_\kappa \left(\frac{e}{mc^2 \kappa^2} \right)^2 \left(\frac{\hbar c \kappa}{2 \epsilon_0 V} \right)$$

$$= \frac{4\pi^2}{V^2} \alpha^2 \left(\frac{\hbar}{mc} \right)^4 \sum_\kappa \frac{1}{\kappa^6}$$

$$= \frac{4\pi^2}{V^2} \alpha^2 \lambda^4 \sum_\kappa \frac{1}{\kappa^6} \quad -(19)$$

and

$$\langle (\delta \underline{r} \cdot \delta \underline{r})^3 \rangle = \left(\frac{2\pi}{V} \alpha \lambda^2 \right)^3 \sum_\kappa \frac{1}{\kappa^9} \quad -(20)$$

using eq. (14):

$$\langle \delta \underline{r} \cdot \delta \underline{r} \rangle = \frac{2}{\pi} \alpha \lambda^2 \int \frac{d\kappa}{\kappa} \quad -(21)$$

$$\langle (\delta \underline{r} \cdot \delta \underline{r})^2 \rangle = \frac{4}{V} (\alpha \lambda^2)^2 \int \frac{d\kappa}{\kappa^4} \quad -(22)$$

$$\langle (\delta \underline{r} \cdot \delta \underline{r})^3 \rangle = \frac{8\pi}{V^2} (\alpha \lambda^2)^3 \int \frac{d\kappa}{\kappa^7} \quad -(23)$$

and so on.

These results may now be implemented with tensor Taylor series for any problem in physics.

For example:

$$\int_{\pi/a_0}^{mc/\hbar} \frac{d\kappa}{\kappa^4} d\kappa = -\frac{1}{3\kappa^3} \Big|_{\pi/a_0}^{mc/\hbar} \quad -(24)$$

$$= \frac{10^{-30}}{3} \left(\frac{1}{5.936^3} - \frac{1}{6.570^3} \right)$$

and so on.

Note carefully that $\langle (\delta \underline{r} \cdot \delta \underline{r})^2 \rangle$ is inversely proportional to the radiation volume V, and that $\langle (\delta \underline{r} \cdot \delta \underline{r})^3 \rangle$ is inversely proportional to V^2. However, $\langle \delta \underline{r} \cdot \delta \underline{r} \rangle$ does not depend on the radiation volume. For a small radiation volume, higher order terms become important.