94(7): Evaluation of the Contact Term

The contact term is:

$$\underline{B} = -\frac{\mu_0}{4\pi} \underline{m} \nabla^2 \left(\frac{1}{r} \right) \quad -(1)$$

This can be evaluated in terms of the Dirac delta function:

$$\underline{B} = \mu_0 \underline{m} \, \delta_D(\underline{r}), \quad -(2)$$

and the effect of the vacuum is:

$$\underline{B} \rightarrow \mu_0 \underline{m} \, \delta_D(\underline{r} + \delta \underline{r}) \quad -(3)$$

There are many expressions in mathematics for the Dirac delta function, and each could be adapted for the replacement:

$$\underline{r} \longrightarrow \underline{r} + \delta \underline{r} \quad -(4)$$

This will give many interesting results in hyperfine structure theory.

However, an initial calculation can be made from first principles using:

$$\nabla^2 \left(\frac{1}{r} \right) = \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2} \right) \frac{1}{r} \quad -(5)$$

where

$$r^2 = X^2 + Y^2 + Z^2. \quad -(6)$$

Consider a term such as:

$$\frac{\partial f}{\partial X} = \frac{\partial f}{\partial r} \frac{\partial r}{\partial X} \quad ; \quad f = \frac{1}{r} \quad -(7)$$

It follows that:

$$\frac{\partial f}{\partial X} = -\frac{1}{r^2}\frac{\partial}{\partial X}\left(X^2+Y^2+Z^2\right)^{1/2} \quad -(8)$$

So:

$$\frac{\partial}{\partial X}\frac{1}{r} = -\frac{X}{r^3}. \quad -(9)$$

Therefore:

$$\frac{\partial^2}{\partial X^2}\frac{1}{r} = \frac{\partial}{\partial X}\frac{\partial r}{\partial X} = -\frac{1}{r^3} - X\frac{\partial}{\partial X}\left(\frac{1}{r^3}\right) \quad -(10)$$

Now use:

$$\frac{\partial}{\partial X}\left(\frac{1}{r^3}\right) = \frac{\partial}{\partial r}\left(\frac{1}{r^3}\right)\frac{\partial r}{\partial X} = -\frac{3}{r^4}\frac{\partial r}{\partial X} \quad -(11).$$

where

$$\frac{\partial r}{\partial X} = \frac{\partial}{\partial X}\left(X^2+Y^2+Z^2\right)^{1/2}$$

$$= \frac{X}{\left(X^2+Y^2+Z^2\right)^{1/2}} = \frac{X}{r} \quad -(12)$$

It follows that:

$$\frac{\partial^2}{\partial X^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3X^2}{r^5} \quad -(13)$$

and

$$\nabla^2\left(\frac{1}{r}\right) = \left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}+\frac{\partial^2}{\partial Z^2}\right)\frac{1}{r} \quad -(14)$$

$$= -\frac{1}{r^3} + \frac{3\left(X^2+Y^2+Z^2\right)}{r^5}$$

This has a similar structure to the magnetic

3) dipole field:

$$\underline{B} = \frac{\mu_0}{4\pi r^3}\left(3\underline{m}\cdot\frac{\underline{r}\,\underline{r}}{r^2} - \underline{m}\right) \quad -(15)$$

i.e. $$\underline{B}_c = -\frac{\mu_0}{4\pi}\underline{m}\,\nabla^2\left(\frac{1}{r}\right)$$

$$= -\frac{\mu_0}{4\pi}\underline{m}\left(3\frac{\underline{r}\cdot\underline{r}}{r^5} - \frac{1}{r^3}\right) \quad -(16)$$

$$= \mu_0\underline{m}\,\delta_D(\underline{r})$$

The effect of the vacuum is to produce the novel contact term:

$$\underline{B}_c = -\frac{\mu_0}{4\pi}\underline{m}\left(\frac{3(\underline{r}+\delta\underline{r})\cdot(\underline{r}+\delta\underline{r})}{|\underline{r}+\delta\underline{r}|^5} - \frac{1}{|\underline{r}+\delta\underline{r}|^3}\right) \quad -(17)$$