90(2): Complete Solution for Precessing Orbits

In this case it is assumed that the scalar potential is, to an excellent approximation:

$$\Phi = -\frac{MG}{(X^2+Y^2)^{1/2}}, \quad -(1)$$

because precessional effects are small.

It follows as in Note 389(4) that for forward precession:

$$\omega_X = \frac{X}{X^2+Y^2}\left(\frac{1}{\gamma}-1\right) - \frac{\dot{X}\dot{Y}Y + X\dot{X}^2}{\gamma c^2(X^2+Y^2)} \quad -(2)$$

$$\omega_Y = \frac{Y}{X^2+Y^2}\left(\frac{1}{\gamma}-1\right) - \frac{\dot{Y}\dot{X}X + Y\dot{Y}^2}{\gamma c^2(X^2+Y^2)} \quad -(3)$$

and for retrograde precession:

$$\omega_X = \left(\frac{X}{X^2+Y^2}\right)\left(\left(1-\frac{\dot{X}^2+\dot{Y}^2}{c^2}\right)^{3/2}-1\right) \quad -(4)$$

$$\omega_X = \left(\frac{Y}{X^2+Y^2}\right)\left(\left(1-\frac{\dot{X}^2+\dot{Y}^2}{c^2}\right)^{3/2}-1\right) \quad -(5)$$

where

$$\gamma = \left(1-\frac{\dot{X}^2+\dot{Y}^2}{c^2}\right)^{-1/2} \quad -(6)$$

The spin connections have been found from conservation of scalar antisymmetry.

From conservation of vector antisymmetry:

$$\frac{\partial Q_Z}{\partial Y} + \frac{\partial Q_Y}{\partial Z} = \omega_Y Q_Z + \omega_Z Q_Y \quad -(7)$$

$$\frac{\partial Q_X}{\partial Z} + \frac{\partial Q_Z}{\partial X} = \omega_Z Q_X + \omega_X Q_Z \quad -(8)$$

$$\frac{\partial Q_Y}{\partial X} + \frac{\partial Q_X}{\partial Y} = \omega_X Q_Y + \omega_Y Q_X \quad -(9)$$

These have particular solutions:

$$\left.\begin{aligned} \frac{\partial A_Z}{\partial Y} &= \omega_Y A_Z, & \frac{\partial A_Y}{\partial Z} &= \omega_Z A_Y, \\ \frac{\partial A_X}{\partial Z} &= \omega_Z A_X, & \frac{\partial A_Z}{\partial X} &= \omega_X A_Z, \\ \frac{\partial A_Y}{\partial X} &= \omega_X A_Y, & \frac{\partial A_X}{\partial Y} &= \omega_Y A_X \end{aligned}\right] \quad -(10)$$

in which:

$$\omega_Z = 0. \quad -(11)$$

Therefore:

$$\left.\begin{aligned} \frac{\partial A_Z}{\partial Y} &= \omega_Y A_Z, & \frac{\partial A_Y}{\partial Z} &= 0, \\ \frac{\partial A_X}{\partial Z} &= 0, & \frac{\partial A_Z}{\partial X} &= \omega_X A_Z, \\ \frac{\partial A_Y}{\partial X} &= \omega_X A_Y, & \frac{\partial A_X}{\partial Y} &= \omega_Y A_X \end{aligned}\right] \quad -(12)$$

The general solution of the first order differential eqn.:

$$\frac{dy}{dx} + y\,p(x) = q(x) \quad -(13)$$

is

$$y = \frac{1}{u(x)}\left(\int u(x)\,q(x)\,dx + C\right) \quad -(14)$$

where

$$u(x) = \exp\left(\int p(x)\,dx\right). \quad -(15)$$

If:

$$q(x) = 0 \quad -(16)$$

then

$$\frac{dy}{dx} = -y\,p(x) \quad -(17)$$

whose solution is:

$$y = C \exp\left(-\int p(x)\, dx\right) \quad -(18)$$

$$Q_X = Q_X(0) \exp\left(\int \omega_Y\, dY\right) \quad -(19)$$

$$Q_Y = Q_Y(0) \exp\left(\int \omega_X\, dX\right) \quad -(20)$$

$$Q_Z = Q_Z(0) \exp\left(\int \omega_Y\, dY\right) \quad -(21)$$

$$= Q_Z(0) \exp\left(\int \omega_X\, dX\right)$$

which:

$$\frac{\partial Q_X}{\partial Z} = \frac{\partial Q_Y}{\partial Z} = 0 \quad -(22)$$

The vector potential is:

$$\underline{Q} = Q_X \underline{i} + Q_Y \underline{j} + Q_Z \underline{k} \quad -(23)$$

In general however:

$$\int \omega_Y\, dY \neq \int \omega_X\, dX \quad -(24)$$

so from eq. (21):

$$Q_Z(0) = 0 \quad -(25)$$

is the only self consistent solution. So:

$$Q_Z = 0 \quad -(26)$$

and

$$\underline{Q} = Q_X \underline{i} + Q_Y \underline{j} \quad -(27)$$

The gravitomagnetic field is found from $\underline{Q}$ and

:

4)

$$\underline{\Omega} = \underline{\nabla} \times \underline{Q} - \underline{\omega} \times \underline{Q} \quad - (28)$$

The scalar component of the spin connection is found from conservation of trace antisymmetry (the Lindstrom constraint):

$$\frac{1}{c^2}\left(\frac{\partial}{\partial t} + \omega_0\right)\underline{\Phi} = \left(\underline{\nabla} - \underline{\omega}\right) \cdot \underline{Q} \quad - (29)$$

From eq. (1):

$$\frac{\partial \underline{\Phi}}{\partial t} = 0 \quad - (30)$$

so

$$\omega_0 = \frac{c^2}{\underline{\Phi}}\left(\underline{\nabla} - \underline{\omega}\right) \cdot \underline{Q} \quad - (31)$$

The scalar spin connection must also satisfy the scalar law of conservation of antisymmetry:

$$\underline{g} = -\underline{\nabla}\,\underline{\Phi} + \underline{\omega}\,\underline{\Phi} = -\frac{\partial \underline{Q}}{\partial t} - \omega_0 \underline{Q} \quad - (32)$$

If it is assumed that:

$$\frac{\partial \underline{Q}}{\partial t} = \underline{0} \quad - (33)$$

then

$$\underline{g} = -\omega_0 \underline{Q} \quad - (34)$$

i.e.

$$\left(g_X^2 + g_Y^2\right)^{1/2} = -\omega_0\left(Q_X^2 + Q_Y^2\right)^{1/2}$$

$$- (35)$$

If ω_0 from eq. (35) is not the same as ω_0

from eq. (31), then regauge $\underline{\Phi}$ and $\underline{Q}$:

$$\underline{\Phi} \rightarrow \underline{\Phi} + \frac{\partial \phi}{\partial t} \quad -(36)$$

where ϕ is the gauge function.

The method of regauging $\underline{Q}$ is to define:

$$\underline{g} := -\underline{\nabla}\,\underline{\Phi} - \frac{\partial \underline{Q}}{\partial t}\,(\text{total}) \quad -(37)$$

$$\underline{\Omega} = \underline{\nabla} \times \underline{Q}\,(\text{total}) \quad -(38)$$

where

$$\underline{Q}\,(\text{total}) = \underline{Q} + \underline{Q}_1 \quad -(39)$$

and

$$\underline{\nabla} \times \underline{Q}_1 = -\underline{\omega} \times \underline{Q} \quad -(40)$$

The four potential to be regauged is defined as:

$$Q^\mu(\text{total}) := \left(\frac{\underline{\Phi}}{c}, \underline{Q}_{\text{total}} \right) \quad -(41)$$

and the regauging is:

$$\underline{\Phi} \rightarrow \underline{\Phi} + \frac{\partial \phi}{\partial t} \quad -(42)$$

$$\underline{Q}_{\text{total}} \rightarrow \underline{Q}_{\text{total}} - \underline{\nabla}\,\phi \quad -(43)$$

under which the observable $\underline{g}$ and $\underline{\Omega}$ are unchanged.

Eqs. (37) and (38) satisfy the ECE2 homogeneous field equations of gravitation:

$$\underline{\nabla} \cdot \underline{\Omega} = 0$$

$$\underline{\nabla} \times \underline{g} + \frac{\partial \underline{\Omega}}{\partial t} = \underline{0} \quad -(44)$$

) By definition:

$$-\frac{\partial \underline{Q}(total)}{\partial t} = \underline{\omega}\,\Phi. \quad -(45)$$

Regauging this equation means that: $\quad -(46)$

$$-\frac{\partial}{\partial t}\left(\underline{Q}(total) - \underline{\nabla}\phi\right) = \underline{\omega}\left(\Phi + \frac{\partial \phi}{\partial t}\right).$$

From eqs. (45) and (46):

$$\frac{\partial}{\partial t}\underline{\nabla}\phi = \underline{\omega}\frac{\partial \phi}{\partial t} \quad -(47)$$

i.e

$$\underline{\nabla}\frac{\partial \phi}{\partial t} = \underline{\omega}\frac{\partial \phi}{\partial t} \quad -(48)$$

Also, note that:

$$\underline{Q}(total) = -\int \underline{\omega}\,\Phi\, dt = -\underline{\omega}\int \Phi\, dt \quad -(49)$$

If

$$\Phi \rightarrow \Phi + \frac{\partial \phi}{\partial t} \quad -(50)$$

then

$$\underline{Q}(total) \rightarrow -\int \underline{\omega}\,\Phi\, dt - \phi\,\underline{\omega} \quad -(51)$$

$$= \underline{Q}(total) - \underline{\nabla}\phi$$

so

$$\boxed{\underline{\nabla}\phi = \underline{\omega}\,\phi} \quad -(52)$$

Regauging means that:

$$\boxed{\begin{aligned} &\underline{Q}(total) \rightarrow \underline{Q}(total) - \underline{\omega}\,\phi \quad &-(53)\\ &\Phi \rightarrow \Phi + \frac{\partial \phi}{\partial t} \quad &-(54)\end{aligned}}$$

By definition:

$$\underline{Q}(\text{total}) = \underline{Q} + \underline{Q}_1 \quad -(55)$$

and

$$\underline{\nabla} \times \underline{Q}_1 = -\underline{\omega} \times \underline{Q} \quad -(56)$$

so

$$\underline{Q} + \underline{Q}_1 \rightarrow \underline{Q} + \underline{Q}_1 - \underline{\omega}\phi \quad -(57)$$

Since ϕ is arbitrary, it can always be assumed that:

$$\underline{Q} \rightarrow \underline{Q} - \underline{\omega}\phi \quad -(58)$$

This means that:

$$\underline{\nabla} \times \underline{Q}_1 \rightarrow \underline{\nabla} \times \underline{Q}_1 - \underline{\omega} \times \underline{\omega}\phi_1 \quad -(59)$$

$$= \underline{\nabla} \times \underline{Q}_1$$

Also:

$$\underline{Q}_1 \rightarrow \underline{Q}_1 - \underline{\omega}\phi \quad -(60)$$

Therefore the gauge transformation is:

$$\boxed{\begin{aligned} \underline{\Phi} &\rightarrow \underline{\Phi} + \frac{\partial \phi}{\partial t} \quad -(61) \\ \underline{Q} &\rightarrow \underline{Q} - \underline{\omega}\phi \quad -(62) \\ \underline{\nabla}\phi &= \underline{\omega}\phi \quad -(63) \end{aligned}}$$

and

From eq. (63):

$$\frac{\partial \phi}{\partial X} = \omega_X \phi \quad -(64)$$

$$\frac{\partial \phi}{\partial Y} = \omega_Y \phi \quad -(65)$$

So $$\phi = \phi_1(0) \exp\left(\int \omega_X \, dX\right) \quad - (66)$$
$$= \phi_2(0) \exp\left(\int \omega_Y \, dY\right)$$
Here $\phi_1(0)$ and $\phi_2(0)$ are constants of integration.

From eq. (62):
$$a_X \rightarrow a_X - \omega_X \phi \quad -(67)$$
$$a_Y \rightarrow a_Y - \omega_Y \phi \quad -(68)$$
so eq. (35) is rearranged to:
$$\left(g_X^2 + g_Y^2\right)^{1/2} = -\omega_0 \left(a_X^2 + a_Y^2\right)^{1/2} \quad -(69)$$
$$\rightarrow -\omega_0 \left(\left(a_X - \omega_X \phi\right)^2 + \left(a_Y - \omega_Y \phi\right)^2\right)$$
and ϕ is adjusted so that ω_0 from eq. (69) is the same as ω_0 from eq. (31).

Finally, from eq. (2) of Note 390(1), the following equation must be obeyed:
$$\underline{\nabla} \cdot \left(\underline{\omega} \, \underline{\Phi}\right) = 0 \quad -(70)$$
This is obeyed in the Newtonian limit, where:
$$\omega_X = \omega_Y \rightarrow 0 \quad -(71)$$
in eqs. (2) and (3). More generally, eqs. (1) to (3) and (70) must be solved numerically.