390(5): General Methodology for ECE2 Orbits

In general, orbits are three dimensional, so the general $\underline{\Phi}$ and $\underline{Q}$ are found from:

$$\square \underline{\Phi} = 4\pi G \rho_m \quad -(1)$$

$$\square \underline{Q} = \frac{4\pi G}{c^2} \underline{J}_m \quad -(2)$$

where ρ_m is the source mass density and where $\underline{J}_m$ is the source current of mass density. Here $\underline{\Phi}$ is the scalar potential and $\underline{Q}$ is the vector potential of gravitation.

The precession of planar orbits is observed experimentally to be very tiny, so to an excellent approximation:

$$\underline{\Phi} = -\frac{2MG}{r} \quad -(3)$$

Note carefully that:

$$r = r(t). \quad -(4)$$

For example in an ellipse, r changes with time.

Therefore the ECE2 Hamiltonian and lagrangian are:

$$H = \gamma mc^2 - \frac{mG}{r} \quad -(5)$$

$$\mathcal{L} = -\frac{mc^2}{\gamma} + \frac{mG}{r} \quad -(6)$$

In 3-D:

$$r = (x^2 + y^2 + z^2)^{1/2} \quad -(7)$$

and the Lorentz factor is:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(8)$$

in which

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \quad -(9)$$

Apply the three Euler Lagrange equations with lagrangian (6),

2) and Lagrange variables X, Y and Z. This leads to forward and retrograde precession in 3-D, and solving the Euler Lagrange equations also gives:

$$\underline{g} = \ddot{X}\,\underline{i} + \ddot{Y}\,\underline{j} + \ddot{Z}\,\underline{k} \quad -(10)$$

The spin connection vector is found from:

$$\underline{g} = -\underline{\nabla}\,\underline{\Phi} + \underline{\omega}\,\Phi \quad -(11)$$

because $\underline{g}$, $\underline{\Phi}$ and $\underline{\nabla}\,\underline{\Phi}$ are known.

The vector potential $\underline{Q}$ is found from the vector antisymmetry law, ensuring automatically that vector antisymmetry is conserved.

In these equations:

$$\underline{\omega} = \omega_X\,\underline{i} + \omega_Y\,\underline{j} + \omega_Z\,\underline{k} \quad -(12)$$

and

$$\underline{Q} = Q_X\,\underline{i} + Q_Y\,\underline{j} + Q_Z\,\underline{k} \quad -(13)$$

with

$$\underline{\Phi} = -\frac{mG}{(X^2+Y^2+Z^2)^{1/2}} \quad -(14)$$

Having found $\underline{\omega}$ and $\underline{Q}$, the scalar spin connection is found from the conservation of trace antisymmetry (Lindstrom constraint):

$$\frac{1}{c^2}\left(\frac{d}{dt}+\omega_0\right)\underline{\Phi} = \left(\underline{\nabla}-\underline{\omega}\right)\cdot\underline{Q} \quad -(15)$$

The map of the vacuum, aether or spacetime is given by the spin connection four vector:

$$\omega^{\mu} = \left(\frac{\omega_0}{c}, \boldsymbol{\omega} \right) \quad -(16)$$

and this can be graphed for 3-D forward and retrograde precession.

The law of conservation of scalar antisymmetry is

$$\boldsymbol{g} = -\boldsymbol{\nabla}\Phi + \boldsymbol{\omega}\Phi = -\frac{\partial \boldsymbol{Q}}{\partial t} - \omega_0 \boldsymbol{Q} \quad -(17)$$

and this can be used to find the function $\frac{\partial \boldsymbol{Q}}{\partial t}$. Note carefully that in general:

$$\frac{\partial \Phi}{\partial t} = \frac{d\Phi}{dt} \neq 0 \quad -(18)$$

even at the Newtonian level. So in general:

$$\boxed{\frac{\partial \boldsymbol{Q}}{\partial t} \neq \boldsymbol{0}} \quad -(19)$$

With this procedure, self consistency between the trace and scalar antisymmetry is achieved. The need for regauging is eliminated.

Finally simultaneous solution of the wave equation (1) and the field equation:

$$\boldsymbol{\nabla} \cdot \boldsymbol{g} = 4\pi G \rho_m \quad -(20)$$

gives the equation:

$$\frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = \frac{1}{c^2}\frac{d^2 \Phi}{dt^2} = \boldsymbol{\nabla} \cdot \left(\boldsymbol{\omega} \Phi \right) \quad -(21)$$

7) In the classical limit:

$$c \to \infty, \ \underline{\omega} \to \underline{0} \quad -(22)$$

So

$$\gamma \to 1 \quad -(23)$$

It follows that eq. (21) is always true in the classical limit, even though

$$\frac{d^2 \Phi}{dt^2} (\text{Newton}) \neq 0 \quad -(24)$$

In ECE2 relativity, c is the speed of light, a universal constant, so eq. (21) gives the function

$$\frac{d^2 \Phi}{dt^2} = - \frac{d^2}{dt^2} \left(\frac{mG}{r(t)} \right) \quad -(25)$$

<u>Calculation of $\partial^2 \Phi / \partial t^2$</u> in <u>Newtonian Gravitation</u>.

In this case:

$$\frac{1}{r(t)} = \frac{1}{\alpha} \left(1 + \epsilon \cos \phi(t) \right) \quad -(26)$$

and

$$\frac{\partial}{\partial t} \left(\frac{1}{r(t)} \right) = \frac{d}{dt} \left(\frac{1}{r(t)} \right) \quad -(27)$$

$$= - \frac{\epsilon}{\alpha} \omega(t) \sin \phi(t)$$

where the angular ~~momentum~~ velocity is:

$$\omega = \frac{d\phi}{dt} = \frac{L}{\mu r^2} \quad -(28)$$

where L is the constant angular momentum and

$$\mu \sim \frac{mM}{m+M} \ \text{if} \ M \gg m \quad -(29)$$

5) From eq. (27) it follows that:

$$\frac{d^2}{dt^2}\left(\frac{1}{r(t)}\right) = -\frac{\epsilon}{\alpha}\left(\frac{d\omega}{dt}\sin\phi + \omega\frac{d\sin\phi(t)}{dt}\right) \quad -(30)$$

$$= -\frac{\epsilon}{\alpha}\left(\frac{d\omega}{dt}\sin\phi + \omega\frac{d\phi}{dt}\cos\phi\right)$$

$$= -\frac{\epsilon}{\alpha}\left(\frac{d\omega}{dt}\sin\phi + \omega^2\cos\phi\right)$$

Here:

$$\omega = \frac{L}{mr^2} = \frac{L}{m\alpha^2}\left(1+\epsilon\cos\phi(t)\right)^2 \quad -(31)$$

So

$$\frac{\partial^2 \Phi}{\partial t^2} = \frac{d^2 \Phi}{dt^2} = \frac{mG\epsilon}{\alpha}\left(\frac{d\omega}{dt}\sin\phi + \frac{L^2}{m^2\alpha^4}\left(1+\epsilon\cos\phi\right)^4\right) \quad -(32)$$

in which:

$$\frac{d\omega}{dt} = \frac{2L}{m\alpha^2}\left(1+\epsilon\cos\phi\right)\frac{d}{dt}\left(1+\epsilon\cos\phi(t)\right)$$

$$= -\frac{2L\omega}{m\alpha^2}\left(1+\epsilon\cos\phi\right)\epsilon\sin\phi(t)$$

$$= -\frac{2L\omega\epsilon}{m\alpha^2}\left(1+\epsilon\cos\phi\right)\sin\phi(t)$$

$$= -2\epsilon\left(\frac{L}{m\alpha^2}\right)^2\left(1+\epsilon\cos\phi\right)^3\sin\phi \quad -(33)$$

Therefore it follows that $\partial^2\Phi/\partial t^2$ at the Newtonian level is not zero:

$$\frac{d^2\Phi}{dt^2} = \frac{mG\epsilon}{d}\left[\frac{L^2}{m^2d^4}(1+\cos\phi)^4 - 2\epsilon\left(\frac{L}{md^2}\right)^2(1+\epsilon\cos\phi)^3\sin^2\phi\right]$$

$$= \frac{mG\epsilon}{d}\left(\frac{L}{md^2}\right)^2(1+\cos\phi)^3\left(1+\epsilon\cos\phi-2\epsilon\sin^2\phi\right) \quad -(34)$$

Finally, the Newtonian dependence of ϕ on t is found from:

$$\frac{d\phi}{dt} = \frac{L}{mr^2} \quad -(35)$$

where

$$\frac{1}{r^2} = \frac{1}{d^2}(1+\epsilon\cos\phi)^2 \quad -(36)$$

so

$$\frac{d\phi}{dt} = \frac{L}{md^2}(1+\epsilon\cos\phi)^2 \quad -(37)$$

and

$$\frac{dt}{d\phi} = \frac{md^2}{L(1+\epsilon\cos\phi)^2}. \quad -(38)$$

It follows that:

$$t = \frac{md^2}{L}\int\frac{d\phi}{(1+\epsilon\cos\phi)^2} \quad -(39)$$

The integral must be evaluated numerically, because the analytical solution is very complicated, and given by:

$$\int \frac{dx}{(a+b\cos x)^2} = \frac{b\sin x}{(b^2-a^2)(a+b\cos x)} - \frac{a}{b^2-a^2}\int \frac{dx}{a+b\cos x}, \quad -(40)$$

here

$$\int \frac{dx}{a+b\cos x} = \frac{2}{(a^2-b^2)^{1/2}} \tan^{-1}\left(\frac{a\tan(x/2)+b}{(a^2-b^2)^{1/2}}\right) \quad -(41)$$

if

$$a^2 < b^2$$

and

$$\int \frac{dx}{a+b\cos x} = \frac{2}{(a^2-b^2)^{1/2}} \tan^{-1}\left(\frac{(a-b)\tan(x/2)}{(a^2-b^2)^{1/2}}\right) \quad -(42)$$

if

$$a^2 > b^2.$$

Here

$$a = 1, \quad b = \epsilon \quad -(43)$$

so eq. (42) is used for the ellipse and eq. (41) for the hyperbola.

After finding $t(\phi)$ it must be inverted to give $\phi(t)$. This procedure is used in Newtonian astronomy. For small ϵ:

$$\phi(t) = \frac{2\pi t}{\tau} + 2\epsilon \sin\frac{2\pi t}{\tau} + \frac{5}{4}\epsilon^2 \sin\frac{4\pi t}{\tau} \quad -(44)$$
$$+ \frac{1}{12}\epsilon^3\left(13\sin\frac{6\pi t}{\tau} - 3\sin\frac{2\pi t}{\tau}\right) + \ldots$$

So the Newtonian $\partial^2\Phi/\partial t^2$ is found from eq. (34) and (44), and is not zero.

A very important consequence of this fact is that the Newtonian Poisson equation:

$$\nabla^2 \Phi = -4\pi G \rho_m \quad -(45)$$

is incomplete. It must be replaced by the Newtonian d'Alembert wave equation:

$$\left(\frac{1}{v^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \Phi_w = -4\pi G \rho_m \quad -(46)$$

where v is the Newtonian orbital velocity:

$$v = MG \left(\frac{2}{r} - \frac{1}{a} \right) \quad -(47)$$

where a is the semi major axis:

$$a = \frac{\alpha}{1-\epsilon^2} \quad -(48)$$

From eq. (46), propagating gravitational waves exist at the Newtonian level. In eq. (46), Φ_w is the wave's scalar potential. It is different from the inverse square (3), and similar to the Lienard Wiechert potentials of electrodynamics.

If the result:

$$\frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = \underbrace{\nabla \cdot \left(\underline{\omega} \Phi \right)}_{=0} \quad -(49)$$

is found for a precessing orbit, it must be developed as:

$$\frac{\partial^2}{\partial t^2} \Phi(r(t)) = -MG \frac{\partial^2}{\partial t^2} \left(\frac{1}{r(t)} \right) = 0 \quad -(50)$$

giving the dependence of r and t for the orbit.