S1(3): The Spin Cyclic Theorem

The well known $\underline{B}^{(3)}$ field of ECE2 unified field theory is defined by:

$$\underline{B}^{(3)*} = -\frac{i\kappa}{A^{(0)}}\,\underline{A} \times \underline{A}^* \qquad -(1)$$

$$= -\frac{i\kappa}{A^{(0)}}\,\underline{A}^{(1)} \times \underline{A}^{(2)}$$

where

$$\underline{A} \times \underline{A}^* = \underline{A}^{(1)} \times \underline{A}^{(2)} \qquad -(2)$$

is the conjugate product of vector potentials, notably plane waves. For example, if:

$$\underline{A}^{(1)} = \frac{A^{(0)}}{\sqrt{2}}\left(\underline{i} - i\underline{j}\right) e^{i(\omega t - \kappa Z)} \qquad -(3)$$

then

$$\underline{A}^{(2)} = \frac{A^{(0)}}{\sqrt{2}}\left(\underline{i} + i\underline{j}\right) e^{-i(\omega t - \kappa Z)} \qquad -(4)$$

So

$$\underline{A}^{(1)} \times \underline{A}^{(2)} = \frac{A^{(0)2}}{2}\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -i & 0 \\ 1 & i & 0 \end{vmatrix} \qquad -(5)$$

$$= i A^{(0)2}\,\underline{k}$$

So

$$\underline{B}^{(3)*} = \kappa A^{(0)}\,\underline{k}$$

$$= B^{(0)}\,\underline{k} \qquad -(6)$$

$$= B^{(0)}\,\underline{e}^{(3)*}$$

The B Cyclic Theorem follows:

$$\mathbf{B}^{(1)} \times \mathbf{B}^{(2)} = i B^{(0)} \mathbf{B}^{(3)*} \quad -(7)$$

$$\mathbf{B}^{(3)} \times \mathbf{B}^{(1)} = i B^{(0)} \mathbf{B}^{(2)*} \quad -(8)$$

$$\mathbf{B}^{(2)} \times \mathbf{B}^{(3)} = i B^{(0)} \mathbf{B}^{(1)*} \quad -(9)$$

in the complex circular basis:

$$\mathbf{e}^{(1)} \times \mathbf{e}^{(2)} = i \mathbf{e}^{(3)*} \quad -(10)$$

$$\mathbf{e}^{(3)} \times \mathbf{e}^{(1)} = i \mathbf{e}^{(2)*} \quad -(11)$$

$$\mathbf{e}^{(2)} \times \mathbf{e}^{(3)} = i \mathbf{e}^{(1)*} \quad -(12)$$

Here

$$\mathbf{e}^{(1)} = \frac{1}{\sqrt{2}} (\mathbf{i} - i \mathbf{j}) \quad -(13)$$

$$\mathbf{e}^{(2)} = \frac{1}{\sqrt{2}} (\mathbf{i} + i \mathbf{j}) \quad -(14)$$

$$\mathbf{e}^{(3)} = \mathbf{k} \quad -(15)$$

where the Cartesian basis is:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \quad -(16)$$

$$\mathbf{k} \times \mathbf{i} = \mathbf{j} \quad -(17)$$

$$\mathbf{j} \times \mathbf{k} = \mathbf{i} \quad -(18)$$

In ECE2 electrodynamics:

$$\mathbf{B} = \nabla \times \mathbf{A} - \boldsymbol{\omega} \times \mathbf{A} \quad -(19)$$

where $\boldsymbol{\omega}$ is the spin connection vector. Here:

$$\mathbf{B}\,(\text{interaction with vacuum}) = -\boldsymbol{\omega} \times \mathbf{A} \quad -(20)$$

It follows that:

$$\underline{B}^{(3)*} = -i\,\underline{B}^{(1)} \times \underline{B}^{(2)} = -i\frac{\kappa}{A^{(0)}}\,\underline{A}^{(1)} \times \underline{A}^{(2)}$$

$$= -i\,\underline{\omega}^{(1)} \times \underline{A}^{(2)} \quad -(21)$$

So the spin connection is:

$$\underline{\omega}^{(1)} = \frac{\kappa}{A^{(0)}}\,\underline{A}^{(1)} \quad -(22)$$

Similarly:

$$\underline{\omega}^{(2)} = \frac{\kappa}{A^{(0)}}\,\underline{A}^{(2)} \quad -(23)$$

It follows that:

$$\boxed{\underline{B}^{(3)*} = -i\frac{A^{(0)}}{\kappa}\,\underline{\omega}^{(1)} \times \underline{\omega}^{(2)} = -i\frac{B^{(0)}}{\kappa^2}\,\underline{\omega}^{(1)} \times \underline{\omega}^{(2)}} \quad -(24)$$

Note carefully that

$$\boxed{\underline{A}^{(3)} = \underline{\omega}^{(3)} = \underline{0}} \quad -(25)$$

because polar vectors cannot be obtained from a conjugate product by symmetry.

Now define the magnetization of the vacuum by :

4) $$\underline{B}^{(3)*} = \frac{1}{\mu_0}\underline{M}^{(3)*} = -\frac{i\kappa}{A^{(0)}}\underline{A}\times\underline{A}^* = -i\underline{\omega}\times\underline{A}^* \quad -(26)$$

where
$$\underline{\omega} = \frac{\omega^{(0)}}{\sqrt{2}}(\underline{i} - i\underline{j})e^{i\phi} \quad -(27)$$

and
$$\underline{A}^* = \frac{A^{(0)}}{\sqrt{2}}(\underline{i} + i\underline{j})e^{-i\phi} \quad -(28)$$

In eq. (26):
$$\underline{\omega}\times\underline{A}^* = \frac{\omega^{(0)}A^{(0)}}{2}\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -i & 0 \\ 1 & i & 0 \end{vmatrix} \quad -(29)$$
$$= i\omega^{(0)}A^{(0)}.$$

Note that:
$$\omega_x A_y^* = -\omega_y A_x^* \quad -(30)$$

and
$$\omega_Z = A_Z^* = 0 \quad -(31)$$

Eqs. (30) and (31) are solutions of the antisymmetry equations:

$$\frac{\partial A_Z^*}{\partial Y} + \frac{\partial A_Y^*}{\partial Z} = \omega_Y A_Z^* + \omega_Z A_Y^* \quad -(32)$$
$$\frac{\partial A_X^*}{\partial Z} + \frac{\partial A_Z^*}{\partial X} = \omega_Z A_X^* + \omega_X A_Z^* \quad -(33)$$
$$\frac{\partial A_Y^*}{\partial X} + \frac{\partial A_X^*}{\partial Y} = \omega_X A_Y^* + \omega_Y A_X^* \quad -(34)$$

Eqs. (32) to (34) reduce to:

$$\frac{\partial A_y^*}{\partial Z} = \omega_Z A^* \quad -(35)$$

$$\frac{\partial A_x^*}{\partial Z} = \omega_Z A_x^* \quad -(36)$$

$$\omega_x A_y^* = -\omega_y A_x^* \quad -(37)$$

in which:

$$A_x^* = \frac{A^{(0)}}{\sqrt{2}} e^{-i(\omega t - \kappa Z)} \quad -(38)$$

$$A_y^* = -\frac{iA^{(0)}}{\sqrt{2}} e^{-i(\omega t - \kappa Z)} \quad -(39)$$

From eq. (38):

$$\frac{\partial A_x^*}{\partial Z} = i\kappa A_x^* \quad -(40)$$

From eq. (39)

$$\frac{\partial A_y^*}{\partial Z} = \kappa A_x^* \quad -(41)$$

so

$$\omega_Z = i\kappa = \kappa \quad -(42)$$

The only possible solution is

$$\omega_Z = 0 \quad -(43)$$

which is eq. (31) Q.E.D.

Therefore the system rigorously obeys the conservation of antisymmetry.

6) The vector potential is:

$$\underline{A}^* = \underline{A}^{(2)} = \frac{A^{(0)}}{\sqrt{2}}\left(\underline{i} + i\underline{j}\right) e^{-i\phi} \quad -(44)$$

and the spin connection vector is:

$$\underline{\omega} = \underline{\omega}^{(1)} = \frac{\kappa}{\sqrt{2}}\left(\underline{i} - i\underline{j}\right) e^{i\phi} \quad -(45)$$

Eq. (24) is named the spin cyclic theorem.

The other antisymmetry equations are:

$$\underline{E} = -\underline{\nabla}\phi + \underline{\omega}\phi = -\frac{\partial \underline{A}}{\partial t} - \omega_0 \underline{A} \quad -(46)$$

and

$$\frac{1}{c^2}\left(\frac{\partial}{\partial t} + \omega_0\right)\phi = \left(\underline{\nabla} - \underline{\omega}\right)\cdot\underline{A} \quad -(47)$$

For the longitudinal components:

$$\underline{E}^{(3)} = \underline{\omega}^{(3)} = \underline{A}^{(3)} = \underline{0} \quad -(48)$$

so in this case:

$$\underline{\nabla}\phi = \underline{0}, \quad -(49)$$

and

$$\left(\frac{\partial}{\partial t} + \omega_0\right)\phi = 0, \quad -(50)$$

a possible solution of which is:

$$\phi^{(3)} = 0 \quad -(51)$$

For the transverse components:

7) $$\underline{A}^* = \underline{A}^{(2)} = \frac{A^{(0)}}{\sqrt{2}}(\underline{i} + i\underline{j})e^{-i\phi} \quad -(52)$$

So:
$$\underline{E}^{(2)} = -\frac{\partial \underline{A}^{(2)}}{\partial t} - \omega_0 \underline{A}^{(2)} \quad -(53)$$

$$= -\frac{i\omega A^{(0)}}{\sqrt{2}}(\underline{i} + i\underline{j})e^{-i\phi} - \omega_0 \underline{A}^{(2)}$$

$$= -\frac{i\omega A^{(0)}}{\sqrt{2}}(\underline{i} + i\underline{j})e^{-i\phi} - \frac{\omega_0 A^{(0)}}{\sqrt{2}}(\underline{i} + i\underline{j})e^{-i\phi}$$

$$= \frac{\omega A^{(0)}}{\sqrt{2}}(-i\underline{i} + \underline{j})e^{-i\phi} - \frac{\omega_0 A^{(0)}}{\sqrt{2}}(\underline{i} + i\underline{j})e^{-i\phi}.$$

In general, ω_0 may be found from:
$$\underline{B}^{(2)} = \underline{\nabla} \times \underline{A}^{(2)} \quad -(54)$$

and
$$\underline{\nabla} \times \underline{E}^{(2)} + \frac{\partial \underline{B}^{(2)}}{\partial t} = \underline{0} \quad -(55)$$

This is best done by computer ~~algebra~~.
The Lindstrom constraint for $\underline{A}^{(2)}$ is:
$$\frac{1}{c^2}\left(\frac{\partial}{\partial t} + \omega_0\right)\phi = (\underline{\nabla} - \underline{\omega}) \cdot \underline{A}^* \quad -(56)$$

in which:
$$\underline{\nabla} \cdot \underline{A}^* = 0 \quad -(57)$$

and:

$$\underline{\omega} \cdot \underline{A}^* = \frac{\omega^{(0)} A^{(0)}}{2} (\underline{i} - i\underline{j}) \cdot (\underline{i} + i\underline{j})$$

$$= \omega^{(0)} A^{(0)} \quad -(58)$$

So

$$\frac{1}{c^2}\left(\frac{\partial}{\partial t} + \omega_0\right)\phi = \omega^{(0)} A^{(0)} = \kappa A^{(0)} \quad -(59)$$

and ϕ may be found.

Another procedure is to evaluate ϕ^* from:

$$\underline{\nabla} \cdot \underline{E}^* = \Box \phi^* = 0 \quad -(60)$$

for the transverse plane wave $\underline{E}^*$. So:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi^* = 0 \quad -(61)$$

$$-(62)$$

A solution is:

$$\boxed{\phi^* = \phi^{(0)} \exp\left(-i\left(\omega t - \kappa Z\right)\right)}$$

with

$$\kappa = \omega c \quad -(63)$$

Find ω_0 from

$$\frac{1}{c^2}\left(\frac{\partial}{\partial t} + \omega_0\right)\phi = \kappa A^{(0)} \quad -(64)$$

Now adapt the procedure of Note 388(6), and define:

$$\underline{E} = -\underline{\nabla}\phi + \underline{\omega}\phi := -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t} \text{ (total)} \quad -(65)$$

So: $$-\frac{\partial \underline{A}(\text{total})}{\partial t} = \underline{\omega}\,\phi \quad - (66)$$

where $$\underline{A}(\text{total}) = \underline{A}^* + \underline{A}_1^* \quad - (67)$$

and: $$\underline{\nabla} \times \underline{A}_1^* = -\underline{\omega} \times \underline{A}^* \quad - (68)$$

So $$\underline{B}^* = \underline{\nabla} \times \underline{A}(\text{total}) \quad - (69)$$

From eqs. (65) and (69):
$$\underline{\nabla} \times \underline{E}^* + \frac{\partial \underline{B}^*}{\partial t} = \underline{0} \quad - (70)$$

From eq. (66):
$$\underline{A}^*(\text{total}) = -\int \underline{\omega}\,\phi^*\,dt + \underline{A}_2^* \quad - (71)$$

so $$\underline{A}_2^* = \underline{A}(\text{total}) + \int \underline{\omega}\,\phi^*\,dt \quad - (72)$$

This procedure effectively solves eqs. (46) and (47) simultaneously so all CG antisymmetry laws are conserved.