Note 389(6): Final Version of Spin Connection from Note 389(5)

Co author Horst Eckardt evaluated the vector spin connection from Note 389(5) by solving the corresponding equations by computer to give:

$$\underline{\omega} = \underline{\omega}^{(1)} = \frac{\kappa}{\sqrt{2}}(\underline{i} - i\underline{j})e^{i\phi_0} - i\kappa\underline{k} \quad -(1)$$

$$\underline{\omega}^{*} = \underline{\omega}^{(2)} = \frac{\kappa}{\sqrt{2}}(\underline{i} + i\underline{j})e^{-i\phi_0} + i\kappa\underline{k} \quad -(2)$$

Note that the longitudinal component is pure imaginary, so its real and physical part is zero. This is consistent with the fact that the conjugate product is:

$$\underline{\omega}^{(1)} \times \underline{\omega}^{(2)} = \left[\frac{\kappa}{\sqrt{2}}(\underline{i} - i\underline{j})e^{i\phi_0} - i\kappa\underline{k}\right] \times \left[\frac{\kappa}{\sqrt{2}}(\underline{i} + i\underline{j})e^{-i\phi_0} + i\kappa\underline{k}\right]. \quad -(3)$$

The longitudinal part of eq. (3) gives the $\underline{B}^{(3)}$ field:

$$\underline{B}^{(3)*} = \kappa A^{(0)}\underline{k} = B^{(0)}\underline{k} \quad -(4)$$

$$= -i\frac{A^{(0)}}{\kappa}\underline{\omega}^{(1)} \times \underline{\omega}^{(2)}$$

in which:

$$\underline{\omega}^{(1)} = \underline{\omega}^{(2)*} = \frac{\kappa}{\sqrt{2}}(\underline{i} - i\underline{j})e^{i\phi_0}, \quad -(5)$$

$$\phi_0 := \omega t - \kappa Z. \quad -(5a)$$

Respgde

$$\underline{A} = \frac{A^{(0)}}{\sqrt{2}} \left(\underline{i} - i\underline{j} \right) e^{i\phi_0} \quad -(6)$$

$$\underline{A}^* = \frac{A^{(0)}}{\sqrt{2}} \left(\underline{i} + i\underline{j} \right) e^{-i\phi_0} \quad -(7)$$

$$\underline{\omega} = \frac{\kappa}{\sqrt{2}} \left(\underline{i} - i\underline{j} \right) e^{i\phi_0} - i\kappa \underline{k} \quad -(8)$$

$$\underline{\omega}^* = \frac{\kappa}{\sqrt{2}} \left(\underline{i} + i\underline{j} \right) e^{-i\phi_0} + i\kappa \underline{k} \quad -(9)$$

The Lindstrom constraint is:

$$\frac{1}{c^2} \left(\frac{\partial}{\partial t} + \omega_0 \right) \phi^* = \left(\underline{\nabla} - \underline{\omega} \right) \cdot \underline{A}^* \quad -(10)$$

which is defined self consistently with:

$$\frac{\partial A_Z^*}{\partial Y} + \frac{\partial A_Y^*}{\partial Z} = \omega_Y A_Z^* + \omega_Z A_Y^* \quad -(11)$$

$$\frac{\partial A_X^*}{\partial Z} + \frac{\partial A_Z^*}{\partial X} = \omega_Z A_X^* + \omega_X A_Z^* \quad -(12)$$

$$\frac{\partial A_Y^*}{\partial X} + \frac{\partial A_X^*}{\partial Y} = \omega_X A_Y^* + \omega_Y A_X^* \quad -(13)$$

From eqs. (7) and (8), eq. (10) is:

$$\left(\frac{\partial}{\partial t} + \omega_0 \right) \phi^* = -c^2 \underline{\omega} \cdot \underline{A}^*$$

$$= -c^2 \kappa A^{(0)} \quad -(14)$$

$$= -c^2 B^{(0)}$$

3) Eq. (14) must be solved simultaneously with

$$\underline{E}^* = -\underline{\nabla}\phi^* + \underline{\omega}\,\phi^* = -\frac{\partial \underline{A}^*}{\partial t} - \omega_0 \underline{A}^* \quad -(15)$$

For plane waves, $\underline{E}^*$ is known, so ϕ^* can be found from eq. (15), $\underline{\omega}$ being defined by eq. (1)

In order that:

$$\underline{\nabla}\cdot\underline{B}^* = 0 \quad -(16)$$

and

$$\underline{\nabla}\times\underline{E}^* + \frac{\partial \underline{B}^*}{\partial t} = \underline{0} \quad -(17)$$

define:

$$\underline{E}^* = -\underline{\nabla}\phi^* - \frac{\partial \underline{A}^*(\text{total})}{\partial t} \quad -(18)$$

$$\underline{B} = \underline{\nabla}\times\underline{A}^*(\text{total}) \quad -(19)$$

where

$$\underline{A}^*(\text{total}) = \underline{A}^* + \underline{A}_1^* \quad -(20)$$

and

$$\underline{\nabla}\times\underline{A}_1^* := -\underline{\omega}\times\underline{A}^* \quad -(21)$$

so

$$-\frac{\partial \underline{A}^*(\text{total})}{\partial t} = \underline{\omega}\,\phi^* \quad -(22)$$

and

$$\underline{A}^*(\text{total}) = -\int \underline{\omega}\,\phi^*\,dt + \underline{A}_2^* \quad -(23)$$

where $\underline{A}_2^*$ is a constant potential of integration.

Having calculated ϕ^* from:

$$\underline{E}^* = -\underline{\nabla}\phi^* + \underline{\omega}\,\phi^* \quad -(24)$$

It is possible to calculate ω_0 from eq (14):

$$\left(\frac{\partial}{\partial t} + \omega_0\right)\phi^* = -c^2 B^{(0)} = \text{constant} \quad -(25)$$

Finally the equation:

$$-\underline{\nabla}\phi^* + \underline{\omega}\,\phi^* = -\frac{\partial \underline{A}^*}{\partial t} - \omega_0 \underline{A}^* \quad -(26)$$

must be satisfied with the ϕ^* and ω_0 calculated from eqs. (24) and (25). This is achieved in general by using gauge freedom for $\underline{A}^*$ (total):

$$\underline{A}^*(\text{total}) \rightarrow \underline{A}^*(\text{total}) + \underline{\nabla}\phi^* \quad -(27)$$

This gauge transformation leaves $\underline{B}$ unchanged in eq (19) because:

$$\underline{\nabla} \times \underline{\nabla}\phi^* = \underline{0}. \quad -(28)$$

In general:

$$A^\mu \rightarrow A^\mu + \partial^\mu \chi \quad -(29)$$

is the covariant gauge transform.

Knowing ϕ^*, $\underline{\omega}$, $\underline{A}^*$ and ω_0, the gauge function χ is adjusted so that the anti-symmetry equation (26) is obeyed.