386(1): General Method for Magnetostatics

The general equations for magnetostatics are:

$$\nabla \times \underline{B} = \mu_0 \underline{J} \quad -(1)$$

$$\nabla \cdot \underline{B} = 0 \quad -(2)$$

$$\frac{\partial \underline{B}}{\partial t} = \underline{0} \quad -(3)$$

where $\underline{B}$ is the magnetic flux density and $\underline{J}$ is the current density. Here μ_0 is the vacuum permeability. (1) to (3) and on the ECE2 level the field equations are

$$\underline{B} = \nabla \times \underline{A} - \underline{\omega} \times \underline{A} \quad -(4)$$

The antisymmetry equations are:

$$\left(\frac{\partial}{\partial Y} - \omega_Y\right) A_Z = -\left(\frac{\partial}{\partial Z} - \omega_Z\right) A_Y \quad -(5)$$

$$\left(\frac{\partial}{\partial Z} - \omega_Z\right) A_X = -\left(\frac{\partial}{\partial X} - \omega_X\right) A_Z \quad -(6)$$

$$\left(\frac{\partial}{\partial X} - \omega_X\right) A_Y = -\left(\frac{\partial}{\partial Y} - \omega_Y\right) A_X \quad -(7)$$

In general, eqns. (1) to (7) must be solved simultaneously for a given $\underline{J}$.

Consider the change of variable:

$$\nabla \times \underline{A} - \underline{\omega} \times \underline{A} = \nabla \times \underline{\alpha} \quad -(8)$$

It follows that

$$\underline{B} = \nabla \times \underline{\alpha} \quad -(9)$$

and

$$\underline{\alpha} = \frac{\mu_0}{4\pi} \int \frac{\underline{J}(\underline{x}')}{|\underline{x} - \underline{x}'|} d^3x' \quad -(10)$$

For a circular current loop:

$$\underline{J} = -J_\phi \sin\phi' \, \underline{i} + J_\phi \cos\phi' \, \underline{j} \quad -(11)$$

Jackson chapter 5). So:

$$\alpha_\phi(r,\theta) = \frac{\mu_0}{4\pi} 4Ia \int_0^{2\pi} \frac{\cos\phi' \, d\phi'}{(a^2+r^2-2ar\sin\theta\cos\phi')^{1/2}} \quad -(12)$$

Therefore:

$$\begin{aligned} \underline{\alpha} &= \alpha_\phi \, \underline{e}_\phi \\ &= \alpha_\phi \left(-\sin\phi \, \underline{i} + \cos\phi \, \underline{j} \right) \quad -(13) \\ &= \alpha_\phi \left(\frac{-y}{(x^2+y^2)^{1/2}} \underline{i} + \frac{x}{(x^2+y^2)^{1/2}} \underline{j} \right) \end{aligned}$$

It follows that:

$$\begin{aligned} \underline{\nabla} \times \underline{A} - \underline{\omega} \times \underline{A} &= \underline{\nabla} \times \left(\alpha_x \underline{i} + \alpha_y \underline{j} \right) \quad -(14) \\ &= -\frac{\partial \alpha_y}{\partial z} \underline{i} - \frac{\partial \alpha_x}{\partial z} \underline{j} + \left(\frac{\partial \alpha_y}{\partial x} - \frac{\partial \alpha_x}{\partial y} \right) \underline{k} \end{aligned}$$

i.e.:

$$\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} - (\omega_x A_y - \omega_y A_x) = -\frac{\partial \alpha_y}{\partial z} \quad -(15)$$

$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} - (\omega_y A_z - \omega_z A_y) = -\frac{\partial \alpha_x}{\partial z} \quad -(16)$$

$$\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} - (\omega_z A_x - \omega_x A_z) = \frac{\partial \alpha_y}{\partial x} - \frac{\partial \alpha_x}{\partial y} \quad -(17)$$

Eqs. (5), (6), (7), (15), (16) and (17) are six equations in six unknowns: A_x, A_y, A_z, ω_x, ω_y, ω_z.

For a uniformly magnetized sphere:

$$\underline{A}_\phi(\underline{x}) = \frac{\mu_0}{4\pi} M_0 a^2 \int d\Omega' \frac{\sin\theta' \cos\phi'}{|\underline{x} - \underline{x}'|} \quad -(18)$$

So the vector potential $\underline{A}$ and spin connection vector $\underline{\omega}$ can be found. This procedure conserves antisymmetry in general for magnetostatics.

The scalar antisymmetry law is:

$$\underline{E} = -\underline{\nabla}\phi + \underline{\omega}\phi = -\frac{\partial \underline{A}}{\partial t} - \omega_0 \underline{A} \quad -(19)$$

If it is assumed that:

$$\frac{\partial \underline{A}}{\partial t} = \underline{0} \quad -(20)$$

then

$$\underline{E} = -\omega_0 \underline{A} \quad -(21)$$

and

$$\underline{\nabla} \times \underline{E} = \underline{0} \quad -(22)$$

from eq. (3) and the Faraday law of induction.

From the Ampère Law (1):

$$\frac{\partial \underline{E}}{\partial t} = \underline{0}. \quad -(23)$$

So

$$\underline{\nabla} \times (\omega_0 \underline{A}) = \underline{0} \quad -(24)$$

and ω_0 may be found, given $\underline{A}$.