385(2): Conservation of Antisymmetry for the Electric Dipole Field.

The same general methodology is used as in Note 385(1), starting with the first antisymmetry law:

$$\underline{E} = -\underline{\nabla}\phi + \underline{\omega}\phi = -\frac{\partial \underline{A}}{\partial t} - \omega_0 \underline{A} \quad - (1)$$

in the same notation. It is again assumed that:

$$\frac{\partial \underline{A}}{\partial t} = \underline{0} \quad - (2)$$

so

$$\underline{E} = -\omega_0 \underline{A} \quad - (3)$$

where ω_0 is assumed to be the universal:

$$\omega_0 = -\frac{c}{r} \quad - (4)$$

so

$$\underline{E} = \frac{c}{r}\underline{A} \quad - (5)$$

and

$$\underline{A} = \frac{r}{c}\underline{E} \quad - (6)$$

for electrostatics.

The electric field strength $\underline{E}$ due to a dipole moment $\underline{p}$ is:

$$\underline{E}(\underline{r}) = \frac{3\underline{n}(\underline{p}\cdot\underline{n}) - \underline{p}}{4\pi\epsilon_0 |\underline{r} - \underline{r}_0|^3} \quad - (7)$$

where $\underline{n}$ is a unit vector from $\underline{r}_0$ to $\underline{r}$. The dipole moment of a charge distribution is defined as:

$$\underline{p} = \int \underline{x}' \rho(x') \, d^3x' \quad - (8)$$

2) Using computer algebra, the vector potential $\underline{A}$ can be calculated as:

$$\underline{A} = \frac{r}{c}\left(\frac{3\underline{n}(\underline{p}\cdot\underline{n}) - \underline{p}}{4\pi\epsilon_0 |\underline{r} - \underline{r}_0|^3}\right) \quad -(9)$$

Using the vector antisymmetry laws and assuming the absence of a magnetic flux density, it follows as in Note 385(1) that the spin connections can be calculated from:

$$\frac{\partial A_Z}{\partial Y} = \omega_Y A_Z, \quad \frac{\partial A_Y}{\partial Z} = \omega_Z A_Y, \quad -(10)$$

$$\frac{\partial A_X}{\partial Z} = \omega_Z A_X, \quad \frac{\partial A_Z}{\partial X} = \omega_X A_Z,$$

$$\frac{\partial A_Y}{\partial X} = \omega_X A_Y, \quad \frac{\partial A_X}{\partial Y} = \omega_Y A_X$$

The spin connection vector $\underline{\omega}$ from eq. (10) conserves antisymmetry. In general, antisymmetry is conserved for any static electric field in the absence of a magnetic flux density, provided that the spin connection vector is defined by eqns. (10). For the dipole field, Eqs. (8) to (10) can be worked out with computer algebra.

The problem is simplified by considering a dipole

p is the Z axis. In spherical polar coordinates (Jackson, "Classical Electrodynamics"):

$$E_r = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3}, \quad E_\theta = \frac{p\sin\theta}{4\pi\epsilon_0 r^3}, \quad E_\phi = 0 \quad -(11)$$

and

$$\underline{E} = E_r \underline{e}_r + E_\theta \underline{e}_\theta \quad -(12)$$

Switching to Cartesian coordinates:

$$X = r\sin\theta\cos\phi \quad -(13)$$

$$Y = r\sin\theta\sin\phi \quad -(14)$$

$$Z = r\cos\theta \quad -(15)$$

$$r = (X^2 + Y^2 + Z^2)^{1/2} \quad -(16)$$

$$\underline{e}_r = \underline{i}\sin\theta\cos\phi + \underline{j}\sin\theta\cos\phi + \underline{k}\cos\theta. \quad -(17)$$

$$\underline{e}_\phi = -\underline{i}\sin\phi + \underline{j}\cos\phi \quad -(18)$$

So:

$$\underline{e}_r = \frac{\underline{r}}{r} = \frac{1}{r}(X\underline{i} + Y\underline{j} + Z\underline{k}) \quad -(19)$$

Here:

$$\cos\theta = \frac{Z}{r} \quad -(20)$$

and

$$\sin^2\theta + \cos^2\theta = 1. \quad -(21)$$

Furthermore:

$$\sin\phi = \frac{Y}{(X^2+Y^2)^{1/2}}, \quad \cos\phi = \frac{X}{(X^2+Y^2)^{1/2}} \quad -(22)$$

†) So:

$$\underline{E} = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\, \underline{e}_r + \sin\theta\, \underline{e}_\theta\right) \quad -(23)$$

where:

$$\cos\theta = \frac{z}{r} \quad -(24)$$

and:

$$\sin\theta\, \underline{e}_\theta = \sin\theta\cos\theta\cos\phi\, \underline{i} + \sin\theta\cos\theta\sin\phi\, \underline{j} - \sin^2\theta\, \underline{k}$$

$$= \frac{x}{r}\cos\theta\, \underline{i} + \frac{y}{r}\cos\theta\, \underline{j} - \sin^2\theta\, \underline{k}$$

$$= \frac{1}{r^2}\left(xz\, \underline{i} + yz\, \underline{j} - (x^2+y^2)\, \underline{k}\right) \quad -(25)$$

Therefore:

$$\underline{E} = p\left(\frac{2z}{4\pi\epsilon_0 r^4}\, \underline{e}_r + \frac{1}{4\pi\epsilon_0 r^5}\left(xz\,\underline{i} + yz\,\underline{j} - (x^2+y^2)\underline{k}\right)\right)$$

$$= \frac{p}{4\pi\epsilon_0}\left(\frac{2z\left(x\underline{i} + y\underline{j} + z\underline{k}\right)}{r^5} + \frac{1}{4\pi\epsilon_0 r^5}\left(xz\,\underline{i} + yz\,\underline{j} - (x^2+y^2)\underline{k}\right)\right)$$

$$= \frac{p}{4\pi\epsilon_0 r^5}\left(3xz\,\underline{i} + 3xy\,\underline{j} + (2z^2 - x^2 - y^2)\,\underline{k}\right) \quad -(26)$$

The potential is therefore:

5)

$$\underline{A} = \frac{p}{4\pi\epsilon_0 c r^4}\left(3XZ\,\underline{i} + 3XY\,\underline{j} + (2Z^2 - X^2 - Y^2)\,\underline{k}\right) \quad -(27)$$

where:

$$r^4 = (X^2 + Y^2 + Z^2)^2 \quad -(28)$$

Computer algebra can be used to calculate the spin connections from eqs. (10) and (27), or more generally eqs. (7) and (10). These spin connections automatically conserve antisymmetry. -(29)

S.I. Units Check

$$A = J s C^{-1} m^{-1},\ p = Cm,\ \epsilon_0 = J^{-1} C^2 m^{-1},$$

So

$$A = \frac{Cm}{J^{-1} C^2 m^{-1} m s^{-1} m^2} = J s C^{-1} m^{-1}$$

Note carefully that the vector potential (27) and spin connection:

$$\omega^\mu = (\omega_0, \underline{\omega}) \quad -(30)$$

do not exist in the standard model.