5(7): Meaning of the Hodge Dual Definitions in the Standard Model

- In the usual theory:

$$\underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t} \quad -(1)$$

$$\underline{B} = \underline{\nabla} \times \underline{A} \quad -(2)$$

$$\underline{\nabla} \cdot \underline{B} = 0 \quad -(3)$$

$$\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = \underline{0} \quad -(4)$$

$$\underline{\nabla} \cdot \underline{E} = \rho / \epsilon_0 \quad -(5)$$

$$\underline{\nabla} \times \underline{B} + \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} = \mu_0 \underline{J} \quad -(6)$$

Using the Lorenz gauge condition:

$$\underline{\nabla} \cdot \underline{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0 \quad -(7)$$

it follows that

$$\Box A^\mu = \mu_0 J^\mu \quad -(8)$$

where

$$A^\mu = \left(\frac{\phi}{c}, \underline{A} \right), \quad J^\mu = \left(c\rho, \underline{J} \right) \quad -(9)$$

In the Hodge dual theory:

$$\underline{E} = -c\underline{\nabla} \times \underline{A} \quad -(10)$$

$$c\underline{B} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t} \quad -(11)$$

So eqs. (3) to (6) give:

$$\nabla^2 \phi + \frac{\partial}{\partial t}\left(\underline{\nabla} \cdot \underline{A} \right) = 0 \quad -(12)$$

$$-\underline{\nabla} \times (\underline{\nabla} \times \underline{A}) + \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t}\right) = \underline{0} \quad -(13)$$

$$\underline{\nabla} \cdot \underline{E} = 0 \quad -(14)$$

$$\underline{\nabla} \cdot \underline{B} = 0 \quad -(15)$$

Using (7) in (12):

$$\square \phi = 0 \quad -(16)$$

Using

$$-\underline{\nabla} \times (\underline{\nabla} \times \underline{A}) = \nabla^2 \underline{A} - \underline{\nabla}(\underline{\nabla} \cdot \underline{A}) \quad -(17)$$

and the Lorenz condition (7) in eq. (13) gives:

$$\square \underline{A} = \underline{0} \quad -(18)$$

so eqs. (10) and (11) imply the free space d'Alembert equation:

$$\square A^\mu = 0 \quad -(19)$$

and the free space solution:

$$\underline{\nabla} \cdot \underline{B} = 0 \quad -(20)$$

$$\underline{\nabla} \cdot \underline{E} = 0 \quad -(21)$$

$$\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = \underline{0} \quad -(22)$$

$$\underline{\nabla} \times \underline{B} - \frac{1}{c^2}\frac{\partial \underline{E}}{\partial t} = \underline{0} \quad -(23)$$

In the absence of an electric field:

$$\underline{\nabla} \cdot \underline{B} = 0 \quad -(24)$$

$$\underline{\nabla} \times \underline{B} = \underline{0} \quad -(25)$$

$$\frac{\partial \underline{B}}{\partial t} = \underline{0} \quad -(26)$$

3) Eqs. (11) and (25) imply

$$\underline{\nabla} \times \underline{A} = \underline{0} \quad -(27)$$

Extending this analysis to the ECE2 level:

$$c\underline{B} = -\underline{\nabla}\phi - \underline{\omega}\phi = -\frac{\partial \underline{A}}{\partial t} - \omega_0 \underline{A} \quad -(28)$$

implies

$$\underline{\nabla} \times \underline{B} = \underline{0} \quad -(29)$$

$$\underline{\nabla} \times \underline{A} = \underline{0} \quad -(30)$$

and

$$\underline{B} = B_r \underline{e}_r + B_\theta \underline{e}_\theta \quad -(31)$$

where

$$B_r = \frac{\mu_0}{4\pi}\left(I\pi a^2\right)\frac{\cos\theta}{r^3} \quad -(32)$$

$$B_\theta = \frac{\mu_0}{4\pi}\left(I\pi a^2\right)\frac{\sin\theta}{r^3} \quad -(33)$$

so antisymmetry is conserved for regions where

$$\underline{J} = \underline{0} \quad -(34)$$

More generally at the ECE2 level:

$$\underline{\nabla} \times \underline{B} = \mu_0 \underline{J} \quad -(35)$$

$$\underline{\nabla} \cdot \underline{B} = 0 \quad -(36)$$

$$\frac{\partial \underline{B}}{\partial t} = \underline{0} \quad -(37)$$

and:

$$\underline{B} = \underline{\nabla} \times \underline{A} - \underline{\omega} \times \underline{A} \quad -(38)$$

From eqs. (36) and (38): $\underline{\nabla} \cdot (\underline{\omega} \times \underline{A}) = 0 \quad -(39)$

4) The vector antisymmetry laws must be obeyed:

$$\left(\frac{\partial}{\partial y} - \omega_y\right)A_z = -\left(\frac{\partial}{\partial z} - \omega_z\right)A_y \quad -(40)$$

$$\left(\frac{\partial}{\partial z} - \omega_z\right)A_x = -\left(\frac{\partial}{\partial x} - \omega_x\right)A_z \quad -(41)$$

$$\left(\frac{\partial}{\partial x} - \omega_x\right)A_y = -\left(\frac{\partial}{\partial y} - \omega_y\right)A_x \quad -(42)$$

Eq. (35) is:

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{A} - \underline{\omega} \times \underline{A}) = \mu_0 \underline{J} \quad -(43)$$

where

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{A}) = \underline{\nabla}(\underline{\nabla} \cdot \underline{A}) - \nabla^2 \underline{A} \quad -(44)$$

$$\underline{\nabla} \times (\underline{\omega} \times \underline{A}) = \underline{\omega}(\underline{\nabla} \cdot \underline{A}) - (\underline{\nabla} \cdot \underline{\omega})\underline{A} + (\underline{A} \cdot \underline{\nabla})\underline{\omega} - (\underline{\omega} \cdot \underline{\nabla})\underline{A} \quad -(45)$$

As shown in Note 381(3), a possible solution is:

$$\underline{A} = \frac{B^{(0)}}{2}(-y\underline{i} + x\underline{j}) \quad -(46)$$

$$\underline{\omega} = -\left(\frac{1}{x}\underline{i} + \frac{1}{y}\underline{j}\right) \quad -(47)$$

$$\underline{B} = \underline{\nabla} \times \underline{A} - \underline{\omega} \times \underline{A} = 2B^{(0)}\underline{k} \quad -(48)$$

$$\underline{E} = \underline{0} \quad -(49)$$

The general solution would be similar to this solution for the static magnetic field, and must be found by computer. The static magnetic field corresponds to

$$\underline{J} = \underline{0} \quad -(50)$$

5) so corresponds to

$$\underline{\nabla} \times \underline{B} = \underline{0} \quad - (51)$$

$$\underline{\nabla} \cdot \underline{B} = 0 \quad - (52)$$

therefore $\underline{B}$ can be expressed as eq. (28). If:

$$\frac{\partial \underline{A}}{\partial t} = \underline{0} \quad - (53)$$

then

$$c\underline{B} = -\omega_0 \underline{A} \quad - (54)$$

and from eqs. (48) and (46):

$$\omega_0 = 0 \quad - (55)$$

In general:

$$\omega_0 = \frac{mc^2}{2\pi\hbar} \quad - (56)$$

so the static magnetic field solution is associated with

$$m \rightarrow 0 \quad - (57)$$

where m is the mass of the vacuum particle.