380(5): System to be Solved

This consists of the homogeneous field equations of ECE2 gravitation:

$$\boldsymbol{\nabla}\cdot\boldsymbol{\Omega} = 0 \quad -(1)$$

and

$$\boldsymbol{\nabla}\times\mathbf{g} + \frac{\partial \boldsymbol{\Omega}}{\partial t} = \mathbf{0} \quad -(2)$$

and the antisymmetry laws defined by:

$$\boldsymbol{\Omega} = \boldsymbol{\nabla}\times\boldsymbol{\Omega} - \boldsymbol{\omega}\times\boldsymbol{\Omega}. \quad -(3)$$

Here:

$$\mathbf{g} = -\frac{\partial \mathbf{Q}}{\partial t} - \omega_0 \mathbf{Q} \quad -(4)$$

$$= -\boldsymbol{\nabla}\Phi + \boldsymbol{\omega}\Phi$$

Eq. (1) gives:

$$Q_X\left(\frac{\partial \omega_Z}{\partial Y} - \frac{\partial \omega_Y}{\partial Z}\right) + Q_Y\left(\frac{\partial \omega_X}{\partial Z} - \frac{\partial \omega_Z}{\partial X}\right) + Q_Z\left(\frac{\partial \omega_Y}{\partial X} - \frac{\partial \omega_X}{\partial Y}\right)$$

$$= \omega_X\left(\frac{\partial Q_Z}{\partial Y} - \frac{\partial Q_Y}{\partial Z}\right) + \omega_Y\left(\frac{\partial Q_X}{\partial Z} - \frac{\partial Q_Z}{\partial X}\right) + \omega_Z\left(\frac{\partial Q_Y}{\partial X} - \frac{\partial Q_X}{\partial Y}\right) \quad -(5)$$

Eq. (2) gives:

$$\frac{\partial}{\partial t}\left(\boldsymbol{\omega}\times\mathbf{Q}\right) + \boldsymbol{\nabla}\times\left(\omega_0 \mathbf{Q}\right) = \mathbf{0} \quad -(6)$$

The antisymmetry laws are, from note 380(4):

$$\left(\frac{\partial}{\partial Y} - \omega_Y\right) Q_Z = -\left(\frac{\partial}{\partial Z} - \omega_Z\right) Q_Y \quad -(7)$$

$$\left(\frac{\partial}{\partial Z} - \omega_Z\right) Q_X = -\left(\frac{\partial}{\partial X} - \omega_X\right) Q_Z \quad -(8)$$

$$\left(\frac{\partial}{\partial X} - \omega_X\right) Q_Y = -\left(\frac{\partial}{\partial Y} - \omega_Y\right) Q_X \quad -(9)$$

These can be written as:

$$\frac{\partial Q_Z}{\partial Y} + \frac{\partial Q_Y}{\partial Z} = \omega_Z Q_Y + \omega_Y Q_Z \quad - (10)$$

$$\frac{\partial Q_X}{\partial Z} + \frac{\partial Q_Z}{\partial X} = \omega_X Q_Z + \omega_Z Q_X \quad - (11)$$

$$\frac{\partial Q_Y}{\partial X} + \frac{\partial Q_X}{\partial Y} = \omega_X Q_Y + \omega_Y Q_X \quad - (12)$$

Now assume that:

$$\underline{\Omega} \sim \underline{0} \quad - (13)$$

This means that:

$$\underline{\nabla} \times \underline{Q} = \underline{\omega} \times \underline{Q} \quad - (14)$$

i.e

$$\frac{\partial Q_Z}{\partial Y} - \frac{\partial Q_Y}{\partial Z} = \omega_Y Q_Z - \omega_Z Q_Y \quad - (15)$$

$$\frac{\partial Q_X}{\partial Z} - \frac{\partial Q_Z}{\partial X} = \omega_Z Q_X - \omega_X Q_Z \quad - (16)$$

$$\frac{\partial Q_Y}{\partial X} - \frac{\partial Q_X}{\partial Y} = \omega_X Q_Y - \omega_Y Q_X \quad - (17)$$

Add (10) and (15):

$$\frac{\partial Q_Z}{\partial Y} = \omega_Y Q_Z \quad - (18)$$

Add (11) and (16):

$$\frac{\partial Q_X}{\partial Z} = \omega_Z Q_X \quad - (19)$$

Add (12) and (17):

$$\frac{\partial Q_Y}{\partial X} = \omega_X Q_Y \quad - (20)$$

Subtract (15) from (10):

$$\frac{\partial Q_Y}{\partial Z} = \omega_Z Q_Y. \quad -(21)$$

Subtract (16) from (11):

$$\frac{\partial Q_Z}{\partial X} = \omega_X Q_Z \quad -(22)$$

Subtract (17) from (12):

$$\frac{\partial Q_X}{\partial Y} = \omega_Y Q_X \quad -(23)$$

Eqs. (18) to (23) are six equations in six unknowns: $Q_X, Q_Y, Q_Z, \omega_X, \omega_Y, \omega_Z$. So a general solution can be found.

From eqs. (19) and (21):

$$\frac{\partial Q_X}{\partial Z} = \omega_Z Q_X \quad -(24)$$

and

$$\frac{\partial Q_Y}{\partial Z} = \omega_Z Q_Y \quad -(25)$$

so

$$Q_X = Q_Y \quad -(26)$$

If we try the solution:

$$Q_X = Q_Y = i\, Q_0 \exp\left(i(\omega t - \kappa_Z Z)\right) \quad -(27)$$

then

$$\frac{\partial Q_X}{\partial Z} = \kappa_Z Q_X \quad -(28)$$

So

$$\boxed{\omega_Z = \kappa_Z} \quad -(29)$$

) Fm eqs. (18) and (23):

$$\frac{\partial a_Z}{\partial t} = \omega_Y a_Z \quad -(30)$$

$$\frac{\partial a_X}{\partial t} = \omega_Y a_X \quad -(31)$$

So:

$$a_X = a_Z \quad -(32)$$

Fm eqs. (26) and (32):

$$a_X = a_Y = a_Z. \quad -(33)$$

Therefore:

$$\underline{a} = a_X \underline{i} + a_Y \underline{j} + a_Z \underline{k}$$

$$= a_X (\underline{i} + \underline{j} + \underline{k}) \quad -(34)$$

If we try the solution:

$$a_X = a_Y = a_Z = i a_0 \exp(i(\omega t - \kappa_Z Z - \kappa_X X - \kappa_Y Y))$$

$$= i a_0 \exp(i(\omega t - \underline{\kappa} \cdot \underline{r})) \quad -(35)$$

then:

$$\boxed{\begin{aligned} \omega_X &= \kappa_X \\ \omega_Y &= \kappa_Y \\ \omega_Z &= \kappa_Z \end{aligned}} \quad \begin{aligned} &-(36) \\ &-(37) \\ &-(38) \end{aligned}$$

and

$$\boxed{\underline{\omega} = \kappa_X \underline{i} + \kappa_Y \underline{j} + \kappa_Z \underline{k}} \quad -(39)$$

5) From eqs. (36) to (38), using the solution (27):

$$\frac{\partial \omega_X}{\partial Z} = \frac{\partial \omega_X}{\partial Y} = \frac{\partial \omega_Z}{\partial Y} = \frac{\partial \omega_Y}{\partial Z} = \frac{\partial \omega_Y}{\partial X} = \frac{\partial \omega_Z}{\partial X} = 0 \quad -(40)$$

and
$$\frac{\partial Q_Z}{\partial Y} = \frac{\partial Q_Z}{\partial X} = \frac{\partial Q_Y}{\partial X} = \frac{\partial Q_X}{\partial Y} = 0 \quad -(41)$$

so eq. (5) reduces to:

$$-\omega_X \left(\frac{\partial Q_Y}{\partial Z} \right) + \omega_Y \left(\frac{\partial Q_X}{\partial Z} \right) = 0 \quad -(42)$$

This is true if:

$$\boxed{\omega_X = \omega_Y} \quad -(43)$$

because:
$$Q_X = Q_Y. \quad -(44)$$

Finally, ω_0 is found from eq. (6), which is:

$$\frac{\partial}{\partial t}\left(\omega_Y Q_Z - \omega_Z Q_Y \right) + \omega_0 \left(\frac{\partial Q_Z}{\partial Y} - \frac{\partial Q_Y}{\partial Z} \right) + Q_Z \frac{\partial \omega_0}{\partial Y} - Q_Y \frac{\partial \omega_0}{\partial Z} = 0 \quad -(45)$$

$$\frac{\partial}{\partial t}\left(\omega_Z Q_X - \omega_X Q_Z \right) + \omega_0 \left(\frac{\partial Q_X}{\partial Z} - \frac{\partial Q_Z}{\partial X} \right) + Q_X \frac{\partial \omega_0}{\partial Z} - Q_Z \frac{\partial \omega_0}{\partial X} = 0 \quad -(46)$$

$$\frac{\partial}{\partial t}\left(\omega_X Q_Y - \omega_Y Q_X \right) + \omega_0 \left(\frac{\partial Q_Y}{\partial X} - \frac{\partial Q_X}{\partial Y} \right) + Q_Y \frac{\partial \omega_0}{\partial X} - Q_X \frac{\partial \omega_0}{\partial Y} = 0 \quad -(47)$$

Using the solution (27) in eqs. (22), (23), (30), (31), it is found that:

$$\omega_X = \omega_Y = 0 \quad -(48)$$

) and therefore:

$$\omega_Z = \kappa_Z, \quad \omega_X = \omega_Y = 0 \quad -(49)$$

and:

$$Q_X = Q_Y = Q_Z = i Q_0 \exp\left(i(\omega t - \kappa_Z Z)\right) - (50)$$

Therefore eq. (47) reduces to:

$$0 = 0, \quad -(51)$$

if it is assumed that:

$$\frac{\partial \omega_0}{\partial X} = \frac{\partial \omega_0}{\partial Y} = 0. \quad -(52)$$

If it is further assumed that:

$$\frac{\partial \omega_0}{\partial Z} = 0 \quad -(53)$$

then eq. (45) reduces to:

$$-\frac{\partial}{\partial t}(\omega_Z Q_Y) - \omega_0 \frac{\partial Q_Y}{\partial Z} = 0 \quad -(54)$$

and eq. (46) reduces to:

$$\frac{\partial}{\partial t}(\omega_Z Q_X) + \omega_0 \frac{\partial Q_X}{\partial Z} = 0 \quad -(55)$$

However,

$$Q_X = Q_Y = i Q_0 \exp\left(i(\omega t - \kappa_Z Z)\right) \quad -(56)$$

so eqs. (54) and (55) are the same equation.

From eqs. (55) and (56):

$$\kappa \frac{\partial Q_X}{\partial t} + \omega_0 \frac{\partial Q_X}{\partial Z} = 0 \quad -(57)$$

i.e. $$-\kappa \omega Q_X + \kappa \omega_0 Q_X = 0 \quad -(58)$$

So

$$\boxed{\omega_0 = \omega} \quad - (59)$$

The Complete Solution

$$Q_X = Q_Y = Q_Z = i Q_0 \exp\left(i(\omega t - \kappa_Z Z)\right) \quad - (60)$$

$$\omega^\mu = \left(\frac{\omega}{c}, \underline{\kappa}\right) \quad - (61)$$

and

$$\underline{\Omega} \sim \underline{0} \quad - (62)$$

From eq. (61):

$$\omega^\mu = \kappa^\mu \quad - (63)$$

where κ^μ is the wave four-vector. The energy momentum four vector is:

$$p^\mu = \hbar \kappa^\mu \quad - (64)$$
$$= \hbar \omega^\mu$$

Therefore the energy momentum of a graviton is the spin connection four vector with $\hbar$.

Electromagnetism

In this case the complete solution is:

$$A_X = A_Y = A_Z = i A_0 \exp\left(i(\omega t - \kappa_Z Z)\right) - (65)$$

$$\underline{B} \sim \underline{0} \quad - (66)$$

$$\omega^\mu = \left(\frac{\omega}{c}, \underline{\kappa}\right) \quad - (67)$$

If we wish to consider the case of a finite

) magnetic field it is possible to consider particular solutions of the antisymmetry equations:

$$\frac{\partial A_Z}{\partial Y} + \frac{\partial A_Y}{\partial Z} = \omega_Z A_Y + \omega_Y A_Z \quad - (68)$$

$$\frac{\partial A_X}{\partial Z} + \frac{\partial A_Z}{\partial X} = \omega_X A_Z + \omega_Z A_X \quad - (69)$$

$$\frac{\partial A_Y}{\partial X} + \frac{\partial A_X}{\partial Y} = \omega_X A_Y + \omega_Y A_X \quad - (70)$$

For example:

$$A_X = i A^{(0)} \exp\left(i(\omega t - \kappa Z)\right) \quad - (71)$$

and

$$A_Y = A_Z = 0 \quad - (72)$$

So

$$\underline{A} = A_X \underline{i} \quad - (73)$$

It follows that:

$$\omega^\mu = \left(\frac{\omega}{c}, \underline{\kappa}\right) \quad - (74)$$

and

$$\underline{B} = \underline{\nabla} \times \underline{A} - \underline{\omega} \times \underline{A} \quad - (75)$$

where

$$\underline{\omega} = \omega_Z \underline{k} \quad - (76)$$

So

$$\underline{B} = \left(\frac{\partial A_X}{\partial Z} + \omega_Z A_X\right) \underline{j} \quad - (77)$$

$$= \left(\frac{\partial}{\partial Z} + \omega_Z\right) A_X \, \underline{j}$$

1) So the magnetic field is the covariant derivative of the potential.

From eqns. (71) and (77):

$$\underline{B} = (\kappa_Z + \omega_Z) A_X \underline{j}$$
$$= 2\kappa_Z A_X \underline{j} \quad -(78)$$

i.e

$$\underline{B} = 2i\kappa_Z A^{(0)} \exp(i(\omega t - \kappa Z))\underline{j} \quad -(79)$$

and

$$\boxed{\text{Real } \underline{B} = -2\kappa_Z A^{(0)} \sin(\omega t - \kappa Z)\,\underline{j}}$$

This is a sinusoidal field in the $\underline{j}$ direction.
The electric field strength is:

$$\underline{E} = -\frac{\partial \underline{A}}{\partial t} - \omega_0 \underline{A} \quad -(80)$$
$$= -(\omega + \omega_0)\underline{A}$$

$$\text{Real } \underline{E} = -2\omega_0 A^{(0)} \sin(\omega t - \kappa Z)\,\underline{i} \quad -(81)$$