76(5): Solving for the Orbital and Field Equations

The orbital equation is the Minkowski equation:

$$\underline{F} = \gamma^3 m \underline{g} = -mM\frac{\underline{r}}{r^3} \quad -(1)$$

and the gravitostatic field equations are:

$$\underline{\nabla} \times \underline{g} = \underline{0} \quad -(2)$$

$$\frac{\partial \underline{g}}{\partial t} = \underline{0} \quad -(3)$$

$$\underline{\nabla} \cdot \underline{g} = \underline{\kappa} \cdot \underline{g} = 4\pi MG\rho_m \quad -(4)$$

The Minkowski equation is:

$$\ddot{x} = -MG\left(1 - \frac{\dot{x}^2 + \dot{y}^2}{c^2}\right)^2 \frac{x}{(x^2+y^2)^{3/2}} \quad -(5)$$

$$\ddot{y} = -MG\left(1 - \frac{\dot{x}^2 + \dot{y}^2}{c^2}\right)^2 \frac{y}{(x^2+y^2)^{3/2}} \quad -(6)$$

This can be solved to give a precessing orbit. The gravitational field components are:

$$g_x = \frac{\ddot{x}}{\left(1 + \frac{\dot{x}^2 + \dot{y}^2}{c^2}\right)^2} = -\frac{MGx}{(x^2+y^2)^{3/2}} \quad -(7)$$

$$g_y = \frac{\ddot{y}}{\left(1 + \frac{\dot{x}^2 + \dot{y}^2}{c^2}\right)^2} = -\frac{MGy}{(x^2+y^2)^{3/2}} \quad -(8)$$

2) From eqs. (7) and (8), g_X and g_Y depend on time in general, through $\dot{X}$, $\dot{Y}$, $\ddot{X}$ and $\ddot{Y}$. The field equation:

$$\frac{\partial \underline{g}}{\partial t} = \underline{0} \quad -(9)$$

means that

$$\frac{\partial g_X}{\partial t}\underline{i} + \frac{\partial g_Y}{\partial t}\underline{j} = \underline{0} \quad -(10)$$

Eq. (2) means that:

$$\frac{\partial g_Y}{\partial X} = \frac{\partial g_X}{\partial Y} \quad -(11)$$

and Eq. (4) means that:

$$\frac{\partial g_X}{\partial X} + \frac{\partial g_Y}{\partial Y} = \kappa_X g_X + \kappa_Y g_Y - 4\pi M G \rho_m \quad -(12)$$

The result (11) means that $\underline{g}$ is a central field. The consequences of these field equations can be worked out with computer algebra. Eq. (11) follows from:

$$\frac{\partial g_X}{\partial Y} = \frac{X \frac{\partial}{\partial Y}\left((X^2+Y^2)^{3/2}\right) MG}{(X^2+Y^2)^3} \quad -(13)$$

$$\frac{\partial g_Y}{\partial X} = \frac{Y \frac{\partial}{\partial X}\left((X^2+Y^2)^{3/2}\right) MG}{(X^2+Y^2)^3} \quad -(14)$$

so

$$\frac{\partial g_X}{\partial Y} = \frac{\partial g_Y}{\partial X} \quad -(15)$$

Q.E.D.

3) Eq. (4) can be worked out with:

$$\frac{\partial g_x}{\partial x} = -mG\left(\frac{(x^2+y^2)^{3/2} - \frac{\partial}{\partial x}\left((x^2+y^2)^{3/2}\right)}{(x^2+y^2)^3}\right)$$

$$= -\frac{mG}{(x^2+y^2)^{3/2}}\left(1 - \frac{2x}{(x^2+y^2)^{1/2}}\right) \quad -(16)$$

So

$$\frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} = \frac{-mG}{(x^2+y^2)^{3/2}}\left(2 - \frac{2(x+y)}{(x^2+y^2)^{1/2}}\right)$$

$$= \frac{2mG}{(x^2+y^2)^{3/2}}\left(\frac{x+y}{(x^2+y^2)^{1/2}} - 1\right)$$

$$= \frac{2mG}{(x^2+y^2)^{3/2}}\left(\left(\frac{x+y}{x-y}\right)^{1/2} - 1\right) \quad -(17)$$

$$= 4\pi G \rho_m = \underline{k} \cdot \underline{g}$$

So

$$\boxed{\rho_m = \frac{m}{2\pi(x^2+y^2)^{3/2}}\left(\left(\frac{x+y}{x-y}\right)^{1/2} - 1\right) = \frac{\underline{k} \cdot \underline{g}}{4\pi G}} \quad -(18)$$

where x and y are found from solving eqs. (5) and (6)