376(7): Calculation of the Angular Momentum

Consider the Lagrangian in spherical polar coordinates:

$$\mathcal{L} = -mc^2\left(1-\frac{v^2}{c^2}\right)^{1/2} + \frac{mMG}{r} \quad -(1)$$

where

$$v^2 = \dot{r}^2 + r^2\dot{\phi}^2 \quad -(2)$$

The two relativistic Euler Lagrange equations are:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) \quad -(3)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) \quad -(4)$$

In eq. (3):

$$\frac{p}{m} = \frac{\partial \mathcal{L}}{\partial \dot{r}} = \gamma\dot{r} \quad -(5)$$

In eq. (4):

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \gamma r^2\dot{\phi} \quad -(6)$$

So

$$F = m\frac{dp}{d\tau} = -\frac{mMG}{r^2} \quad -(7)$$

The relativistic angular momentum is:

$$\boxed{L = \gamma m r^2\dot{\phi} = \gamma m r^2\frac{d\phi}{dt}} \quad -(8)$$

From eq. (4):

$$\frac{dL}{d\tau} = \gamma\frac{dL}{dt} = 0 \quad -(9)$$

Eq. (9) is the Minkowski type torque equation.

) The relativistic torque is:

$$Tq = \frac{dL}{d\tau} \quad -(10)$$

and the relativistic angular momentum is:

$$L = \gamma L_0 \quad -(11)$$

where

$$L_0 = mr^2 \frac{d\phi}{dt} \quad -(12)$$

is the non relativistic angular momentum. So:

$$Tq = \frac{dL}{d\tau} = \gamma \frac{dL}{dt} = \gamma \frac{d}{dt}(\gamma L_0) = 0 \quad -(13)$$

In general, the relativistic momentum L does not change w.r.t. proper time:

$$\frac{dL}{d\tau} = 0 \quad -(14)$$

i.e.

$$\gamma \frac{dL}{dt} = 0 \quad -(15)$$

From eq. (13):

$$\frac{dL}{dt} = L_0 \frac{d\gamma}{dt} + \gamma \frac{dL_0}{dt} = 0 \quad -(16)$$

so in general:

$$\frac{dL_0}{dt} \neq 0 \quad -(17)$$

but

$$\frac{dL}{dt} = 0 \quad -(18)$$

as found in a previous paper using Maxima.