371(2): Orbital Precession in Terms of the Euler Angles

The relevant lagrangian is:

$$\mathcal{L} = \frac{1}{2} m \underline{v} \cdot \underline{v} - U(r) \quad -(1)$$

where

$$U(r) = -\frac{mMG}{r} \quad -(2)$$

The frame of reference is defined as $(1, 2, 3)$. The position vector in this frame is:

$$\underline{r} = r_1 \underline{e}_1 + r_2 \underline{e}_2 + r_3 \underline{e}_3 \quad -(3)$$

where $\underline{e}_1$, $\underline{e}_2$ and $\underline{e}_3$ are the unit vectors in frame $(1, 2, 3)$. Note carefully that the axes of this frame are rotating. Therefore the velocity vector in frame $(1, 2, 3)$ is

$$\underline{v} = \frac{d\underline{r}}{dt} + \underline{\omega} \times \underline{r} \quad -(4)$$

where $\underline{\omega}$ is the angular velocity vector:

$$\underline{\omega} = \omega_1 \underline{e}_1 + \omega_2 \underline{e}_2 + \omega_3 \underline{e}_3 \quad -(5)$$

In terms of the Euler angles ϕ, θ and ψ:

$$\omega_1 = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi \quad -(6)$$

$$\omega_2 = \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi \quad -(7)$$

$$\omega_3 = \dot{\phi} \cos\theta + \dot{\psi}. \quad -(8)$$

The velocity vector is:

$$\underline{v} = v_1 \underline{e}_1 + v_2 \underline{e}_2 + v_3 \underline{e}_3 \quad -(9)$$

As shown in immediately preceding papers,

2) Eq (4) in component format is:

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \quad -(10)$$

Therefore:

$$v_1 = \dot{r}_1 - \omega_3 r_2 + \omega_2 r_3 \quad -(11)$$

$$v_2 = \dot{r}_2 + \omega_3 r_1 - \omega_1 r_3 \quad -(12)$$

$$v_3 = \dot{r}_3 - \omega_2 r_1 + \omega_1 r_2 \quad -(13)$$

By definition:

$$r^2 = r_1^2 + r_2^2 + r_3^2 \quad -(14)$$

and

$$\underline{v} \cdot \underline{v} = v^2 = v_1^2 + v_2^2 + v_3^2 \quad -(15)$$

So the lagrangian (1) is:

$$\mathcal{L} = \frac{1}{2} m\left(v_1^2 + v_2^2 + v_3^2\right) - \frac{mMG}{\left(r_1^2 + r_2^2 + r_3^2\right)^{1/2}} \quad -(16)$$

The Lagrange variables are r_1, r_2, r_3, ϕ, θ and χ.

Therefore the orbital motion of m around M is given completely by solving six Euler Lagrange equations in six unknowns:

3)

$$\frac{\partial \mathcal{L}}{\partial r_1} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}_1}\right) \quad -(17)$$

$$\frac{\partial \mathcal{L}}{\partial r_2} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}_2}\right) \quad -(18)$$

$$\frac{\partial \mathcal{L}}{\partial r_3} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}_3}\right) \quad -(19)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) \quad -(20)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) \quad -(21)$$

$$\frac{\partial \mathcal{L}}{\partial \chi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\chi}}\right) \quad -(22)$$

The relation between the spherical polar coordinate angles θ_1 and ϕ_1 and the Euler angles ϕ, θ and ψ is given by:

$$\omega^2 = \dot{\phi}_1{}^2 + \dot{\theta}_1{}^2 = \left(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\right)^2 + \left(\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi\right)^2 + \left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2 \quad -(23)$$

4) By solving eqs. (17) to (22) simultaneously with Maxima, many new orbits emerge. These are defined by $r_1 = r_1(\phi)$; $r_1 = r_1(\theta)$; $r_1 = r_1(\chi)$; $r_2 = r_2(\phi)$; $r_2 = r_2(\theta)$; $r_2 = r_2(\chi)$; $r_3 = r_3(\phi)$; $r_3 = r_3(\theta)$, $r_3 = r_3(\chi)$. There is a total of nine single function orbits. There are also nine double function orbits: $r_1(\phi, \theta)$; $r_1(\phi, \chi)$; $r_1(\theta, \chi)$; $r_2(\phi, \theta)$; $r_2(\phi, \chi)$; $r_2(\theta, \chi)$; $r_3(\phi, \theta)$; $r_3(\phi, \chi)$; $r_3(\theta, \chi)$. Finally there are three triple function orbits: $r_1(\theta, \phi, \chi)$; $r_2(\theta, \phi, \chi)$; $r_3(\theta, \phi, \chi)$.

Therefore Eulerian orbits are very rich in information that can be searched for in astronomy. These are nutations and precessions of $\phi(t)$, $\theta(t)$ and $\chi(t)$.