11(3) : Definition of Reference Frames for Vector $\underline{r}$

The spherical polar system is defined by:

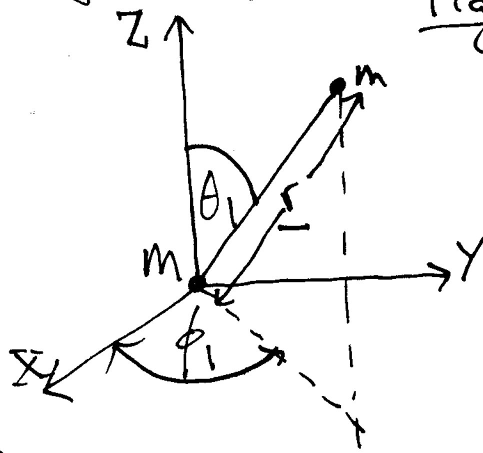

Fig (1)

$$X = r \sin\theta_1 \cos\phi_1 \quad -(1)$$
$$Y = r \sin\theta_1 \sin\phi_1 \quad -(2)$$
$$Z = r \cos\phi_1 \quad -(3)$$
$$\underline{r} = r \underline{e}_r \quad -(4)$$

The mass M is at the origin and the mass m is at the point $\underline{r}$. The vector $\underline{r}$ joins M and m.

Now rotate frame (X, Y, Z) into frame $(1, 2, 3)$

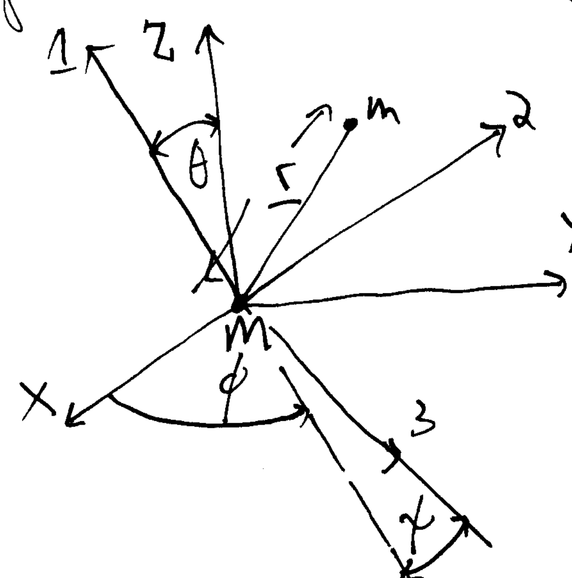

Fig. (2)

The Euler angles are ϕ, θ and χ. In frame $(1, 2, 3)$:

$$\underline{r} = r_1 \underline{e}_1 + r_2 \underline{e}_2 + r_3 \underline{e}_3 \quad -(5)$$

Therefore:

$$\underline{r} = X \underline{i} + Y \underline{j} + Z \underline{k} \quad -(6)$$
$$= r_1 \underline{e}_1 + r_2 \underline{e}_2 + r_3 \underline{e}_3 \quad -(7)$$
$$= r \underline{e}_r \quad -(8)$$

so
$$r^2 = X^2 + Y^2 + Z^2 = r_1^2 + r_2^2 + r_3^2 \quad -(9)$$

Therefore:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \sin\theta_1 \cos\phi_1 & 0 & 0 \\ 0 & \sin\theta_1 \sin\phi_1 & 0 \\ 0 & 0 & \cos\phi_1 \end{bmatrix} \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} \quad - (10)$$

In terms of the Euler angles ϕ, θ and ψ

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \quad - (11)$$

$$= \begin{bmatrix} \sin\theta_1 \cos\phi_1 & 0 & 0 \\ 0 & \sin\theta_1 \sin\phi_1 & 0 \\ 0 & 0 & \cos\phi_1 \end{bmatrix} \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}.$$

It is seen that the use of the Euler angles is an alternative to the use of the spherical polar coordinates. In terms of the spherical polar coordinates the orbital linear velocity is:

$$\underline{v} = \underline{\omega} \times \underline{r} \quad - (12)$$

In component form:

$$\begin{bmatrix} v_r \\ v_\theta \\ v_\phi \end{bmatrix} = \begin{bmatrix} 0 & -\dot{\theta}_1 & -\dot{\phi}_1 \sin\theta_1 \\ \dot{\theta}_1 & 0 & -\dot{\phi}_1 \cos\theta_1 \\ \dot{\phi}_1 \sin\theta_1 & \dot{\phi}_1 \cos\theta_1 & 0 \end{bmatrix} \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} \quad - (13)$$

The total velocity is

$$\underline{v}(\text{total}) = \frac{d\underline{r}}{dt} + \underline{\omega} \times \underline{r} \quad - (14)$$

3) There are similar expressions in terms of the Euler angles as described in Note 371(2).

Denote the matrix A by:

$$A = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -(15)$$

then

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = A^{-1} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad -(16)$$

where A^{-1} is the inverse of A. This can be evaluated by computer to eliminate human error.

Therefore r_1, r_2, and r_3 can be expressed in terms of X, Y, and Z.