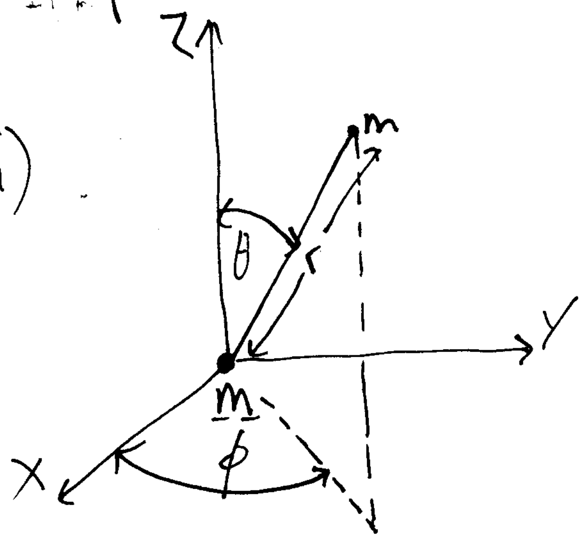
11(3): Scheme for Computation of the Spherical Orbit.

In spherical polar coordinates:



$$X = r\sin\theta\cos\phi \quad -(1)$$
$$Y = r\sin\theta\sin\phi$$
$$Z = r\cos\phi$$

The force of attraction between m and M is:

$$\underline{F} = -\underline{\nabla} U \quad -(2)$$

where

$$U = -\frac{mMG}{r} \quad -(3)$$

The lagrangian is:

$$\mathcal{L} = \frac{1}{2} m\,\underline{v}\cdot\underline{v} - U \quad -(4)$$

where

$$v^2 = \dot{r}^2 + r^2\dot{\beta}^2 \quad -(5)$$

where:

$$\dot{\beta}^2 = \dot{\theta}^2 + \dot{\phi}^2\sin^2\theta \quad -(6)$$

the proper Lagrange variables are r and β, and the Euler Lagrange equations are:

$$\frac{\partial\mathcal{L}}{\partial r} = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right) \quad -(7)$$

and

$$\frac{\partial\mathcal{L}}{\partial\beta} = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\beta}}\right) \quad -(8)$$

These can be solved numerically to give

2) $r(t)$, $\beta(t)$, $\dot{r}(t)$ and $\dot{\beta}(t)$ and

$$\frac{dr}{d\beta} = \frac{dr}{dt}\frac{dt}{d\beta} \quad -(9)$$

It is known from UFT 270 that the orbit is:

$$r = \frac{\alpha}{1+\epsilon\cos\beta} \quad -(10)$$

where α is the half right latitude and ϵ is the eccentricity.

The problem can also be solved with the proper Lagrange variables r, θ and ϕ by simultaneous solution of:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) \quad -(11)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) \quad -(12)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) \quad -(13)$$

to give $r(t)$, $\dot{r}(t)$, $\theta(t)$, $\dot{\theta}(t)$, $\phi(t)$ and $\dot{\phi}(t)$. Therefore $\beta(t)$ can be expressed in terms of $\theta(t)$ and $\phi(t)$. From eq. (8):

$$\dot{\beta}(t) = \frac{L}{mr^2} \quad -(14)$$

where L is a constant of motion. From eqs (10) and (14):

$$\dot{\beta}(t) = \frac{L}{m\alpha^2}\left(1+\epsilon\cos\beta\right)^2 \quad -(15)$$

and this equation can be integrated with Maxima to give $\beta(t)$. The orbit is therefore:

$$r = \frac{\alpha}{1+\epsilon\cos\left(\frac{L}{m\alpha^2}\int\left(1+\epsilon\cos\beta\right)^2 dt\right)} \quad -(16)$$

where

$$\beta(t) = \int\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right)^{1/2} dt \quad -(17)$$

From eq. (13):

$$\dot{\phi} = \frac{L_Z}{mr^2\sin^2\theta} \quad -(18)$$

where L_Z is another constant of motion, the Z component of the angular momentum. As shown in UFT 270 the relevant equations are:

$$m\left(\ddot{r} - r\dot{\beta}^2\right) = -\frac{\partial U}{\partial r} = -\frac{mMG}{r^2} \quad -(19)$$

$$\dot{\beta} = \frac{L}{mr^2} \quad -(20)$$

$$\dot{\phi} = \frac{L_Z}{mr^2\sin^2\theta} \quad -(21)$$

$$\dot{\theta} = \frac{1}{mr^2}\left(L^2 - \frac{L_Z^2}{\sin^2\theta}\right)^{1/2} \quad -(22)$$

$$r = \frac{\alpha}{1+\epsilon\cos\beta} \quad -(23)$$

$$\frac{d\beta}{d\phi} = \frac{L}{L_Z}\sin^2\theta \quad -(24)$$

$$\beta = -\sin^{-1}\left(\frac{L\cos\theta}{\left(L^2-L_Z^2\right)^{1/2}}\right) \quad -(25)$$

$$= \tan^{-1}\left(\frac{L}{L_Z}\tan\phi\right) \quad -(26)$$

$$\phi = \tan^{-1}\left(\frac{L_Z}{L}\tan\beta\right) \quad -(27)$$

$$= -\frac{1}{2}\left(\sin^{-1}\left(\frac{(1+\cos\theta)L^2-L_Z^2}{|1+\cos\theta|\left(L^4-L_Z^2L^2\right)}\right) + \sin^{-1}\left(\frac{(\cos\theta-1)L^2+L_Z^2}{|(\cos\theta-1)|\left(L^4-L_Z^2L^2\right)}\right)\right) \quad -(28)$$

In UFT 276 it was shown that a precessing orbit can be obtained with

$$x = \frac{L}{L_Z} = 1 + \frac{3MG}{c^2\alpha} \quad -(29)$$

using the models:

$$r = \frac{\alpha}{1+\epsilon\cos(x\phi)} \quad -(30)$$

Transition to a planar orbit occurs when:

5)
$$\dot{\theta} \to 0, \quad \theta \to 0 \quad -(31)$$

so the Lagrangian (4) reduces to:

$$\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2 \right) - U \quad -(32)$$

which gives the planar orbit:

$$r = \frac{\alpha}{1 + \epsilon \cos \phi} \quad -(33)$$

This is a static ellipse in the planar limit (31). In three dimensions it may precess.

Scheme for Maxima

Eqs. (19) to (22) are four equations in four unknowns, r, θ, ϕ and β. They can now be solved simultaneously with Maxima to give all the information about any spherical orbit. There is also the additional equation (23), and eq. (6). Therefore $\dot{\theta}$, $\dot{\phi}$ and θ can be found by solving eqs. (19) to (22) simultaneously, and β can be found by integrating eq. (6). Orbit $r(\beta)$ can be found numerically by solving eqs (19) and (20). The orbit $r(\theta)$ can be found from:

$$\frac{L^2}{m^2 r^4} = \dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta \quad -(34)$$

) solved simultaneously with:

$$\dot{\phi} = \frac{L_Z}{mr^2 \sin^2\theta} \quad -(35)$$

and

$$\dot{\theta} = \frac{1}{mr^2}\left(L^2 - \frac{L_Z^2}{\sin^2\theta}\right) \quad -(36)$$

These equations can be solved simultaneously with Maxima to give $r(\beta)$, $r(\theta)$ and $r(\phi)$. No extraneous modelling assumptions are used as in UFT 276. In the limit of the planar orbit, eq. (31) holds,

so

$$\dot{\phi} \to \frac{L_Z}{mr^2}, \quad \dot{\theta} \to 0 \quad -(37)$$

so

$$L \to L_Z \quad -(38)$$

Q.E.D.