70(6): Transformation from Spherical Polar to Eulerian Angles.

Consider the angular velocity in the spherical polar and Eulerian coordinate systems:

$$\underline{\omega} = \omega_r \underline{e}_r + \omega_\theta \underline{e}_\theta + \omega_\phi \underline{e}_\phi$$
$$= \omega_1 \underline{e}_1 + \omega_2 \underline{e}_2 + \omega_3 \underline{e}_3 \qquad -(1)$$

It follows that:

$$\omega_r^2 + \omega_\theta^2 + \omega_\phi^2 = \omega_1^2 + \omega_2^2 + \omega_3^2 \qquad -(2)$$

Denote the spherical polar angles by $\theta_1$ and $\phi_1$ and the Eulerian angle by $\theta$, $\phi$ and $\psi$. So

$$\underline{\omega} = \dot{\phi}_1 \cos\theta_1 \underline{e}_r - \dot{\phi}_1 \sin\theta_1 \underline{e}_\theta + \dot{\theta}_1 \underline{e}_\phi$$
$$= (\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\underline{e}_1 + (\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)\underline{e}_2$$
$$+ (\dot{\phi}\cos\theta + \dot{\psi})\underline{e}_3 \qquad -(3)$$

It follows that:

$$\dot{\phi}_1^2 + \dot{\theta}_1^2 = \dot{\phi}^2\sin^2\theta + \dot{\theta}^2 + (\dot{\phi}\cos\theta + \dot{\psi})^2$$
$$= \dot{\phi}^2 + \dot{\theta}^2 + 2\dot{\phi}\dot{\psi}\cos\theta + \dot{\psi}^2$$

$$\boxed{\dot{\phi}_1^2 + \dot{\theta}_1^2 = \dot{\phi}^2 + \dot{\theta}^2 + \dot{\psi}(2\dot{\phi}\cos\theta + \dot{\psi})} \qquad -(4)$$

In the special case: $\phi_1 = \phi,\ \theta_1 = \theta \qquad -(5)$

then:

$$\dot{\psi} = -2\dot{\phi}\cos\theta \quad \text{or} \quad \dot{\psi} = 0 \qquad -(6)$$