370(4): General Dynamics of the Orbiting Asymmetric Top.

Consider an asymmetric top whose centre of mass is orbiting an object of mass M. The mass of the asymmetric top is m and its principal moments of inertia are I_1, I_2 and I_3. The potential energy of attraction between m and M is considered to be:

$$U = U(\underline{r}, \theta, \phi, \chi) \quad -(1)$$

where $\underline{r}$ is the distance between m and M, and where θ, ϕ and χ are the Euler angles defined in the frame $(1, 2, 3)$ of the principal moments of inertia.

The Lagrangian is:

$$\mathcal{L} = \frac{1}{2} m \underline{\dot{r}} \cdot \underline{\dot{r}} + \frac{1}{2}\left(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2\right) - U(\underline{r}, \theta, \phi, \chi) \quad -(2)$$

So the Lagrange variables are $\underline{r}$, θ, ϕ and χ, and the following four Euler Lagrange equations must be solved simultaneously:

$$\underline{\nabla} \mathcal{L} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \underline{\dot{r}}} \quad -(3)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad -(4)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad -(5)$$

$$\frac{\partial \mathcal{L}}{\partial \chi} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\chi}} \quad -(6)$$

2) Here:

$$\omega_1 = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi \quad -(7)$$

$$\omega_2 = \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi \quad -(8)$$

$$\omega_3 = \dot{\phi} \cos\theta + \dot{\psi} \quad -(9)$$

Eq. (3) gives:

$$m\ddot{\underline{r}} = -\underline{\nabla} U(\underline{r}, \theta, \phi, \psi) \quad -(10)$$

which is the force equation of the orbit.

We have:

$$\omega_1^2 = (\dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi)^2 \quad -(11)$$

$$= \dot{\phi}^2 \sin^2\theta \sin^2\psi + 2\dot{\phi}\dot{\theta} \sin\theta \sin\psi \cos\psi + \dot{\theta}^2 \cos^2\psi$$

$$\omega_2^2 = (\dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi)^2 \quad -(12)$$

$$= \dot{\phi}^2 \sin^2\theta \cos^2\psi - 2\dot{\theta}\dot{\phi} \sin\theta \sin\psi \cos\psi + \dot{\theta}^2 \sin^2\psi$$

$$\omega_3^2 = (\dot{\phi} \cos\theta + \dot{\psi})^2 = \dot{\phi}^2 \cos^2\theta + 2\dot{\phi}\dot{\psi} \cos\theta + \dot{\psi}^2 \quad -(13)$$

So:

$$\mathcal{L} = \frac{1}{2} m \dot{\underline{r}} \cdot \dot{\underline{r}} + \frac{1}{2}(I_1 + I_2)(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{1}{2} I_3 (\dot{\phi}^2 \cos^2\theta + 2\dot{\phi}\dot{\psi}\cos\theta + \dot{\psi}^2)$$

$$+ \dot{\phi}\dot{\theta} \sin\theta \sin\psi \cos\psi \, (I_1 - I_2) - U(\underline{r}, \theta, \phi, \psi) \quad -(14)$$

It follows that:

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\frac{\partial U}{\partial \phi} \quad -(15)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = (I_1 + I_2)\dot{\phi}\sin^2\theta + (I_1 - I_2)\dot{\theta}\sin\theta\sin\psi \ldots$$
$$+ I_3\left(\dot{\phi}\cos^2\theta + \dot{\psi}\cos\theta\right) \quad -(16)$$

Eq. (4) gives:

$$\frac{d}{dt}\Big((I_1 + I_2)\dot{\phi}\sin^2\theta + (I_1 - I_2)\dot{\theta}\sin\theta\sin\psi\cos\psi + I_3\cos\theta\left(\dot{\phi}\cos\theta + \dot{\psi}\right)\Big) = -\frac{\partial U}{\partial \phi} \quad -(17)$$

$$= (I_1 + I_2)\frac{d}{dt}\left(\dot{\phi}\sin^2\theta\right) + (I_1 - I_2)\frac{d}{dt}\left(\dot{\theta}\sin\theta\sin\psi\cos\psi\right) + I_3\frac{d}{dt}\left(\cos\theta\left(\dot{\phi}\cos\theta + \dot{\psi}\right)\right)$$

Similarly:

$$\frac{\partial \mathcal{L}}{\partial \psi} = -\frac{\partial U}{\partial \psi} + (I_1 - I_2)\dot{\phi}\dot{\theta}\sin\theta\left(\cos^2\psi - \sin^2\psi\right) \quad -(18)$$

$$\text{and} \quad \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right) \quad -(19)$$

so:

$$I_3\frac{d}{dt}\left(\dot{\phi}\cos\theta + \dot{\psi}\right) - (I_1 - I_2)\dot{\phi}\dot{\theta}\sin\theta\left(\cos^2\psi - \sin^2\psi\right) = -\frac{\partial U}{\partial \psi} \quad -(20)$$

Finally:

$$\frac{\partial \mathcal{L}}{\partial \theta} = (I_1 + I_2)\sin\theta\cos\theta + \dot{\phi}\dot{\theta}\cos\theta\sin\psi\cos\psi\,(I_1 - I_2)$$
$$- I_3\left(\dot{\phi}^2\sin\theta\cos\theta + \dot{\phi}\dot{\psi}\sin\theta\right) - \frac{\partial U}{\partial \theta} \quad -(21)$$

4) and

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = (I_1+I_2)\dot{\theta} + (I_1-I_2)\dot{\phi}\sin\theta\sin\varphi\cos\varphi \quad -(22)$$

So:

$$(I_1+I_2)\ddot{\theta} + (I_1-I_2)\frac{d}{dt}(\dot{\phi}\sin\theta\sin\varphi\cos\varphi)$$
$$-(I_1+I_2)\sin\theta\cos\theta - (I_1-I_2)\dot{\phi}\dot{\theta}\cos\theta\sin\varphi\cos\varphi$$
$$+ I_3(\dot{\phi}^2\sin\theta\cos\theta + \dot{\phi}\dot{\varphi}\sin\theta) = -\frac{\partial U}{\partial \theta} \quad -(23)$$

Therefore eqs. (15), (20), (17) and (23) must be solved simultaneously.

It is clear that the dynamics depend on $\partial U/\partial r$, $\partial U/\partial \theta$, $\partial U/\partial \phi$ and $\partial U/\partial \varphi$. Therefore models for these functions give orbital characteristics via eq. (10), and different patterns of $\theta(t)$, $\phi(t)$ and $\varphi(t)$.

Example of Modelling the Potential.

Assume:

$$\frac{\partial U}{\partial \phi} = \frac{\partial U}{\partial \varphi} = 0 \quad -(24)$$

and

$$U = U(r, \theta) \quad -(25)$$

), it follows that:

$$\frac{\partial U}{\partial \theta} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial \theta} \quad -(26)$$

$$= \frac{\partial r}{\partial \theta} \underline{\nabla} U$$

The assumption that $\underline{r}$ is a function of θ introduces dependence of the nutations and precession on $\underline{r}$. Using

$$|\underline{\nabla} U| = \frac{nmG}{r^2} = \frac{\partial U}{\partial r} \quad -(27)$$

then

$$\frac{\partial U}{\partial \theta} = \frac{nmG}{r^2}\frac{\partial r}{\partial \theta} \quad -(28)$$

and this can be used in Eq. (23). In the plane polar system (r, θ_1) Eq. (10) gives

$$\ddot{r} - r\dot{\theta}_1^2 = -\frac{\partial U}{\partial r} \quad -(29)$$

and

$$r\ddot{\theta}_1 + 2\dot{r}\dot{\theta}_1 = 0 \quad -(30)$$

The complete system consists of eqs. (15), (17), (20), (23), and (28) to (30).
