570(9): Translational Kinetic Energy of a Gyroscope in Terms of the Euler Angles and Spherical Polar Coordinates

The translational velocity $\underline{v}$ is defined as:

$$\underline{v}_{XYZ} = \left(\underline{v} + \underline{\omega} \times \underline{r}\right)_{123} \quad -(1)$$

The angular velocity in terms of the Euler angles is

$$\underline{\omega} = \omega_1 \underline{e}_1 + \omega_2 \underline{e}_2 + \omega_3 \underline{e}_3 \quad -(2)$$

and in terms of the spherical polar coordinates is:

$$\underline{\omega} = \omega_r \underline{e}_r + \omega_\theta \underline{e}_\theta + \omega_\phi \underline{e}_\phi \quad -(3)$$

where:

$$\omega_1 = \dot{\phi}_1 + \dot{\theta}_1 + \dot{\psi}_1 = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi \quad -(4)$$

$$\omega_2 = \dot{\phi}_2 + \dot{\theta}_2 + \dot{\psi}_2 = \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi \quad -(5)$$

$$\omega_3 = \dot{\phi}_3 + \dot{\theta}_3 + \dot{\psi}_3 = \dot{\phi} \cos\theta + \dot{\psi} \quad -(6)$$

and

$$\omega_r = \dot{\phi}_1 \cos\theta_1 \quad -(7)$$

$$\omega_\theta = -\dot{\phi}_1 \sin\theta_1 \quad -(8)$$

$$\omega_\phi = \dot{\theta}_1 \quad -(9)$$

Now use:

$$\underline{v}_{rot} = \underline{\omega} \times \underline{r} \quad -(10)$$

in frame (1, 2, 3), the rotating frame.

2) It follows that

$$V_{rot}^2 = \underline{\omega} \times \underline{r} \cdot \underline{\omega} \times \underline{r} \quad -(11)$$

$$= \omega^2 r^2 - (\underline{\omega} \cdot \underline{r})(\underline{\omega} \cdot \underline{r})$$

where

$$\underline{r} = r \underline{e}_r \quad -(12)$$

In spherical polar coordinates:

$$V_{rot}^2 = r^2 (\omega^2 - \omega_r^2)$$

$$= r^2 (\dot{\phi}_1^2 + \dot{\theta}_1^2 - \dot{\phi}_1^2 \cos^2 \theta_1)$$

$$= r^2 (\dot{\phi}_1^2 (1 - \cos^2 \theta_1) + \dot{\theta}_1^2) \quad -(13)$$

$$= r^2 (\dot{\theta}_1^2 + \dot{\phi}_1^2 \sin^2 \theta_1)$$

To check this result use:

$$\underline{v} = \dot{r} \underline{e}_r + \underline{v}_{rot} \quad -(14)$$

where

$$\underline{v}_{rot} = r \dot{\theta}_1 \underline{e}_\theta + r \dot{\phi}_1 \sin\theta_1 \underline{e}_\phi \quad -(15)$$

so

$$V_{rot}^2 = r^2 (\dot{\theta}_1^2 + \dot{\phi}_1^2 \sin^2 \theta_1) \quad -(16)$$

which is eq. (13), Q.E.D.

The translational kinetic energy in spherical polars is therefore:

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m (v_x^2 + v_y^2 + v_z^2)$$

3)

$$= \frac{1}{2} m \left(\dot{r}^2 + r^2 \left(\dot{\theta}_1^2 + \dot{\phi}_1^2 \sin^2\theta_1 \right) \right)_{123} \quad - (17)$$

If the potential energy in frame (X, Y, Z) is U, then the lagrangian is:

$$\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \left(\dot{\theta}_1^2 + \dot{\phi}_1^2 \sin^2\theta_1 \right) \right) - U \quad - (18)$$

The Lagrange variables are r, θ_1 and ϕ_1, and the following three Euler Lagrange equations must be solved simultaneously:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}} \right) \quad - (19)$$

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} \right) \quad - (20)$$

$$\frac{\partial \mathcal{L}}{\partial \phi_1} = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_1} \right) \quad - (21)$$

If U is the central gravitational potential:

$$U = - \frac{mG}{r} \quad - (22)$$

This procedure gives the nutation and precession of the earth in orbit around the sun.

+) In terms of the Euler angles:

$$V_{rot}^2 = r^2\left(\omega_1^2 + \omega_2^2 + \omega_3^2 - \omega_1^2\right) \quad -(23)$$

$$= r^2\left(\omega_2^2 + \omega_3^2\right)$$

so the Lagrangian is:

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\left(\left(\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi\right)^2 + \left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2\right)\right) - U \quad -(24)$$

The Lagrange variables are r, ϕ, θ and ψ, and there are four simultaneous Euler Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) \quad -(25)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) \quad -(26)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) \quad -(27)$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}}\right) \quad -(28)$$

Eqs. (24) to (28) give all the information needed to study the Milankovitch cycles.