Note 370(7): Motion of Gyroscopes in Terms of Spherical Polar Coordinates.

The complexity of the problem of describing gyroscope motion can be reduced by expressing the rotational kinetic energy in terms of spherical polar coordinates rather than Eulerian angles.

Gyroscope with One Point Fixed.

In the Eulerian representation the lagrangian for the symmetric top with one point fixed is:

$$\mathcal{L} = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right) - mgh\cos\theta$$

$$= \frac{1}{2}\Big(I_1(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)^2 + I_2(\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)^2 + I_3(\dot{\phi}\cos\theta + \dot{\psi})^2\Big) - mgh\cos\theta \quad -(1)$$

There are three Euler Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad -(2)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad -(3)$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\psi}} \quad -(4)$$

which must be solved simultaneously for $\theta(t)$, $\phi(t)$, $\psi(t)$, $\dot{\theta}(t)$, $\dot{\phi}(t)$ and $\dot{\psi}(t)$.

In the spherical polar representation the same lagrangian is:

$$\mathcal{L} = \frac{1}{2}\left(I_1 \dot{\phi}_1^2 \cos^2\theta_1(t) + I_2 \dot{\phi}_1^2 \sin^2\theta_1(t) + I_3 \dot{\theta}_1^2\right) - mgl\cos\theta_1 \quad -(5)$$

and there are only two Euler Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} \quad -(6)$$

$$\frac{\partial \mathcal{L}}{\partial \phi_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_1} \quad -(7)$$

which can be solved to give $\theta_1(t)$, $\phi_1(t)$, $\dot{\theta}_1(t)$ and $\dot{\phi}_1(t)$.

The angles are related by:

$$\dot{\theta}_1^2 + \dot{\phi}_1^2 = \dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}\left(2\dot{\phi}\cos\theta + \dot{\psi}\right) \quad -(8)$$

and this is a check on the solution.

Application of an External Torque

The angular velocity in general is:

$$\underline{\omega} = \omega_1 \underline{e}_1 + \omega_2 \underline{e}_2 + \omega_3 \underline{e}_3 \quad -(9)$$

in frame defined by the principal moments of inertia, frame (1, 2, 3). The angular momentum in this frame is

$$\underline{L} = I_1 \omega_1 \underline{e}_1 + I_2 \omega_2 \underline{e}_2 + I_3 \omega_3 \underline{e}_3$$

and the torque is:

$$\underline{Tq} = \left(\frac{d\underline{L}}{dt} + \underline{\omega} \times \underline{L} \right)_{123} \quad -(11)$$

Therefore any kind of external torque can be applied. These equations produce the Euler equations:

$$Tq_1 = I_1 \dot{\omega}_1 - (I_2 - I_3)\,\omega_2 \omega_3 \quad -(12)$$

$$Tq_2 = I_2 \dot{\omega}_2 - (I_3 - I_1)\,\omega_3 \omega_1 \quad -(13)$$

$$Tq_3 = I_3 \dot{\omega}_3 - (I_1 - I_2)\,\omega_1 \omega_2 \quad -(14)$$

The complete torque is: $-(15)$

$$\underline{Tq} = Tq_1\, \underline{e}_1 + Tq_2\, \underline{e}_2 + Tq_3\, \underline{e}_3$$

and produces the extra energy:

$$T_{rot} = \frac{1}{2} \left(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2 \right) \quad -(16)$$

The extra torque is applied in the laboratory frame (X, Y, Z):

$$\underline{Tq} = Tq_X\, \underline{i} + Tq_Y\, \underline{j} + Tq_Z\, \underline{k} \quad -(17)$$

$$= Tq_1\, \underline{e}_1 + Tq_2\, \underline{e}_2 + Tq_3\, \underline{e}_3 \quad -(18)$$

So:

$$Tq_X^2 + Tq_Y^2 + Tq_Z^2 = Tq_1^2 + Tq_2^2 + Tq_3^2 \quad -(19)$$

) It is easier to solve eqns. (12), (13) and (14) with spherical polar coordinates for frame (1, 2, 3).

The torque in the laboratory frame is:

$$\underline{Tq} = \underline{r} \times \underline{F} \quad -(20)$$

where $\underline{r}$ is the position vector from the origin to the point where the force $\underline{F}$ is applied. The force is:

$$\underline{F} = -\underline{\nabla} U \quad -(21)$$

where U is the potential energy. In the gyroscope with fixed point:

$$U = mgh\cos\theta \quad -(22)$$

$$-(23).$$

Therefore the force is:

$$\underline{F} = -\underline{\nabla} U = -\left(\frac{\partial U}{\partial r} \underline{e}_r + \frac{1}{r} \frac{\partial U}{\partial \theta} \underline{e}_\theta + \frac{1}{r\sin\theta} \frac{\partial U}{\partial \phi} \underline{e}_\phi \right)$$

in spherical polar coordinates.

Therefore:

$$\underline{F} = \frac{mgh}{r} \sin\theta \, \underline{e}_\theta \quad -(24)$$

and the torque is:

$$\underline{Tq} = \underline{r} \times \underline{F} \quad -(25)$$

where

$$\underline{r} = r \, \underline{e}_r \quad -(26)$$

Therefore the torque is:

$$\underline{Tq} = mgh\sin\theta \, \underline{e}_r \times \underline{e}_\theta \quad -(27)$$

In the spherical polar system (UFT270):

$$\underline{e}_\phi \times \underline{e}_r = \underline{e}_\theta$$
$$\underline{e}_\theta \times \underline{e}_\phi = \underline{e}_r \quad -(28)$$
$$\underline{e}_r \times \underline{e}_\theta = \underline{e}_\phi$$

So

$$\underline{Tq} = mgh \sin\theta \, \underline{e}_\phi . \quad -(29)$$

Converting to Cartesian coordinates:

$$\underline{e}_\phi = -\underline{i} \sin\phi + \underline{j} \cos\phi \quad -(30)$$

so the torque needed to produce the potential energy (22) is:

$$\underline{Tq} = mgh \sin\theta \left(-\underline{i} \sin\phi + \underline{j} \cos\phi \right) \quad -(31)$$

By definition:

$$X = h \sin\theta \cos\phi \quad -(32)$$
$$Y = h \sin\theta \sin\phi \quad -(33)$$

so:

$$\underline{Tq} = mg \left(-Y \underline{i} + X \underline{j} \right) \quad -(34)$$

$$= \underline{r} \times \underline{F}$$

and

$$\underline{r} = r \, \underline{e}_r \quad -(35)$$

and

$$\underline{F} = m\underline{g} = -mg \, \underline{k} \quad -(36)$$

Therefore:

6)

$$\underline{Tq} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ X & Y & 0 \\ 0 & 0 & -mg \end{vmatrix} \quad -(37)$$

$$= mg\left(-Y\underline{i} + X\underline{j}\right)$$

Q.E.D.

Note carefully that the force is in the $-\underline{k}$ direction, the force of the earth's gravity on the centre of mass of the gyro.

Therefore the torque balance equation is:

$$\underline{Tq} = mg\left(-Y\underline{i} + X\underline{j}\right) \quad -(38)$$

$$= Tq_1\underline{e}_1 + Tq_2\underline{e}_2 + Tq_3\underline{e}_3$$

where Tq_1, Tq_2 and Tq_3 are given by the (12) – (14).

Euler equations

Eulerian

$$\left.\begin{aligned} \omega_1 &= \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \omega_2 &= \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \omega_3 &= \dot{\phi}\cos\theta + \dot{\psi} \end{aligned}\right] -(39)$$

Spherical Polar

$$\left.\begin{aligned} \omega_1 &= \dot{\phi}_1\cos\theta_1 &&= \omega_r \\ \omega_2 &= -\dot{\phi}_1\sin\theta_1 &&= \omega_\theta \\ \omega_3 &= \dot{\theta}_1 &&= \omega_\phi \end{aligned}\right] -(40)$$