370(2): Gyroscope Subjected to the General External Torque and Force.

In general the external force is:

$$\underline{F} = m \frac{d^2 \underline{r}}{dt^2} = -\underline{\nabla} U \quad -(1)$$

here

$$\underline{r} = r \, \underline{e}_r \quad -(2)$$

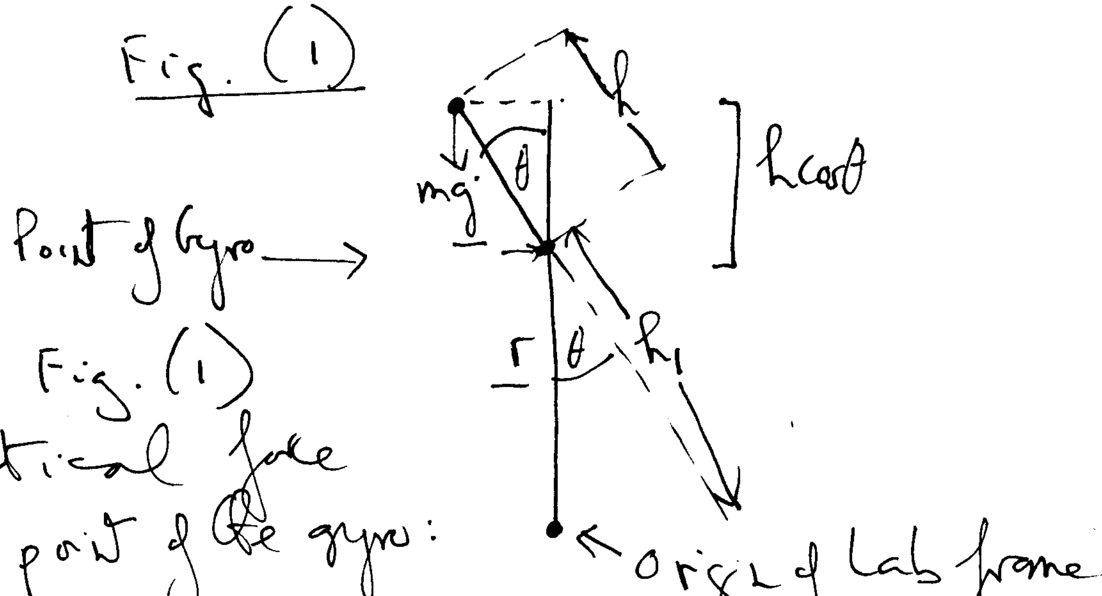
and where the potential energy is:

$$U = U_1 - U_2 = \int_1^2 \underline{F} \cdot \underline{dr}. \quad -(3)$$

The external torque is:

$$\underline{Tq} = \underline{r} \times \underline{F} \quad -(4)$$

Fig. (1)



With respect to Fig. (1) consider a vertical force applied to the point of the gyro:

$$\underline{F} = m \frac{d^2 \underline{r}}{dt^2} \quad -(5)$$

$$:= m \frac{d^2 r}{dt^2} \underline{e}_r = m \ddot{r} \, \underline{e}_r$$

2) This force lifts the point of the gyro and generates the potential energy:

$$U = U_1 - U_2 = \int_1^2 m\ddot{r}\,\underline{e}_r \cdot \underline{r}$$

$$= m\ddot{r}(r_1 - r_2) \quad -(6)$$

$$= -m\ddot{r}\,r$$

where

$$r := \underbrace{r_2 - r_1} \quad -(7)$$

So the lagrangian becomes:

$$\mathcal{L} = \frac{1}{2} m \underline{\dot{r}} \cdot \underline{\dot{r}} + \frac{1}{2} I_{12} \left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2 \right) + \frac{1}{2} I_3 \left(\dot{\phi}\cos\theta + \dot{\psi} \right)^2$$

$$- mgh\cos\theta + m\ddot{r}\,r \quad -(8)$$

in which the velocity of the point of the gyro is:

$$\underline{v} = \underline{\dot{r}} \quad -(9)$$

Now define:

$$r := h_1 \cos\theta \quad -(10)$$

as in Fig (1). It follows that:

$$\dot{r} = -h_1 \left(\dot{\theta} \sin\theta \right) \quad -(11)$$

and

$$\ddot{r} = -h_1 \left(\ddot{\theta} \sin\theta + \dot{\theta}^2 \cos\theta \right) -(12)$$

So:

3)
$$r\ddot{r} = h_1^2\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right)\cos\theta \quad -(13)$$

in which h_1^2 is a constant by construction.

Therefore:

$$\mathcal{L} = \frac{1}{2}m\underline{\dot{r}}\cdot\underline{\dot{r}} + \frac{1}{2}I_{12}\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2$$
$$- mgh\cos\theta + mh_1^2\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right)\cos\theta \quad -(14)$$
$$= \frac{1}{2}m\underline{\dot{r}}\cdot\underline{\dot{r}} + \frac{1}{2}I_{12}\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2$$
$$- mgh\cos\theta + mr\ddot{r} \quad -(14)$$

Eq. (14) means that the force applied to the gyro's point affects the nutations and precession of the gyro. There are four Euler Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) \quad -(15)$$
$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) \quad -(16)$$
$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) \quad -(17)$$
$$\frac{\partial \mathcal{L}}{\partial \psi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}}\right) \quad -(18)$$

Eq. (15) gives:

4)

$$m\ddot{r} = m\ddot{r} \quad -(19)$$

Therefore:

$$\ddot{r} = -h_1\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right) \quad -(20)$$

Eq. (17) gives:

$$\dot{\phi} = \frac{L_\phi - L_\psi\cos\theta}{I_{12}\sin^2\theta} \quad -(21)$$

Eq. (18) gives:

$$\dot{\psi} = \frac{L_\psi}{I_3} - \frac{\left(L_\phi - L_\psi\cos\theta\right)\cos\theta}{I_{12}\sin^2\theta} \quad -(22)$$

and eq. (16) gives:

$$I_{12}\dot{\phi}^2\sin\theta\cos\theta - I_3\left(\dot{\phi}^2\sin\theta\cos\theta + \dot{\phi}\dot{\psi}\sin\theta\right) + mgh\sin\theta$$
$$+ mh_1^2\left(\ddot{\theta}\left(\cos^2\theta - \sin^2\theta\right) - 2\dot{\theta}^2\sin\theta\cos\theta\right)$$
$$= \frac{d}{dt}\left(I_{12}\dot{\theta} + 2mh_1^2\dot{\theta}\cos^2\theta\right)$$
$$= I_{12}\ddot{\theta} + 2mh_1^2\ddot{\theta}\cos^2\theta - 4mh_1^2\dot{\theta}\sin\theta\cos\theta \quad -(23)$$

Subject to checking by computer algebra, the motion of the gyro is found by solving eqs (20) to (23) simultaneously.