370(5): The Free Rotation of an Asymmetric Top in Terms of Spherical Polar Coordinates

In this case the spin connection matrix is:

$$\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\dot\theta & -\dot\phi\sin\theta \\ \dot\theta & 0 & -\dot\phi\cos\theta \\ \dot\phi\sin\theta & \dot\phi\cos\theta & 0 \end{bmatrix} \quad -(1)$$

in the notation of previous papers, so:

$$\omega_1 = \dot\phi\cos\theta \quad -(2)$$

$$\omega_2 = -\dot\phi\sin\theta \quad -(3)$$

$$\omega_3 = \dot\theta \quad -(4)$$

So the angular velocity is:

$$\underline{\omega} = \dot\phi\cos\theta\,\underline{e}_r - \dot\phi\sin\theta\,\underline{e}_\theta + \dot\theta\,\underline{e}_\phi . \quad -(5)$$

For a freely rotating asymmetric top the following is true:

$$\mathcal{L} = T_{rot} = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)$$

$$= \frac{1}{2}\left(I_1\dot\phi^2\cos^2\theta + I_2\dot\phi^2\sin^2\theta + I_3\dot\theta^2\right) \quad -(6)$$

where I_1, I_2 and I_3 are the principal moments of inertia.

There are two Euler Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) \quad -(7)$$

and

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) \quad -(8)$$

Eq. (7) gives:

$$(I_2 - I_1)\frac{d}{dt}\left(\dot{\phi} \sin\theta \cos\theta\right) = 0 \quad -(9)$$

so

$$L_\phi = (I_2 - I_1)\dot{\phi} \sin\theta \cos\theta \quad -(10)$$

is a constant angular momentum of the rotating asymmetric top. It is a constant of motion:

$$\frac{dL_\phi}{dt} = 0. \quad -(11)$$

Note carefully that:

$$\theta = \theta(t) \quad -(12)$$

so:

$$\frac{d}{dt}\left(\dot{\phi}\sin\theta\cos\theta\right) = \ddot{\phi}\sin\theta\cos\theta + \dot{\phi}\frac{d}{dt}\left(\sin\theta\cos\theta\right)$$

$$= \ddot{\phi}\sin\theta\cos\theta + \dot{\phi}\left(\cos\theta\frac{d}{dt}\sin\theta + \sin\theta\frac{d}{dt}\cos\theta\right)$$

$$= \ddot{\phi}\sin\theta\cos\theta + \dot{\phi}\dot{\theta}\left(\cos^2\theta - \sin^2\theta\right) \quad -(13)$$

Therefore Eq. (7) gives:

3) $\ddot{\phi} \sin\theta\cos\theta + \dot{\phi}\dot{\theta}\left(\cos^2\theta - \sin^2\theta\right) = 0$ - (14)

Now note that:

$$\frac{\partial \mathcal{L}}{\partial \theta} = \dot{\phi}^2 \sin\theta\cos\theta\left(I_2 - I_1\right) \quad -(15)$$

and $$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = I_3\ddot{\theta} \quad -(16)$$

so $$I_3\ddot{\theta} = \left(I_2 - I_1\right)\dot{\phi}^2 \sin\theta\cos\theta \quad -(17)$$

Eqs. (14) and (17) can be solved for the trajectories $\theta(t)$ and $\phi(t)$. These give the nutations and processions of the freely rotating asymmetric top in terms of the angles θ and ϕ of the spherical polar coordinate system.

This is a much simpler method than use of the Euler angles.
