370(3): General Theory

Consider the position vector in the moving frame (1, 2, 3):

$$\underline{r} = x_1 \underline{e}_1 + x_2 \underline{e}_2 + x_3 \underline{e}_3 \quad -(1)$$

then

$$\frac{d\underline{r}}{dt} = \dot{x}_1 \underline{e}_1 + \dot{x}_2 \underline{e}_2 + \dot{x}_3 \underline{e}_3 + x_1 \dot{\underline{e}}_1 + x_2 \dot{\underline{e}}_2 + x_3 \dot{\underline{e}}_3 \quad -(2)$$

The same vector $\underline{r}$ in the laboratory frame (X, Y, Z) is

$$\underline{r} = X \underline{i} + Y \underline{j} + Z \underline{k} \quad -(3)$$

and in (X, Y, Z):

$$\left(\frac{d\underline{r}}{dt}\right)_{XYZ} = \dot{X} \underline{i} + \dot{Y} \underline{j} + \dot{Z} \underline{k} \quad -(4)$$

It follows that:

$$\underline{r}_{XYZ} = \underline{r}_{123} \quad -(5)$$

i.e.

$$X \underline{i} + Y \underline{j} + Z \underline{k} = x_1 \underline{e}_1 + x_2 \underline{e}_2 + x_3 \underline{e}_3 \quad -(6)$$

and

$$\left(\frac{d\underline{r}}{dt}\right)_{XYZ} = \left(\frac{d\underline{r}}{dt} + x_1 \dot{\underline{e}}_1 + x_2 \dot{\underline{e}}_2 + x_3 \dot{\underline{e}}_3\right)_{123} \quad -(7)$$

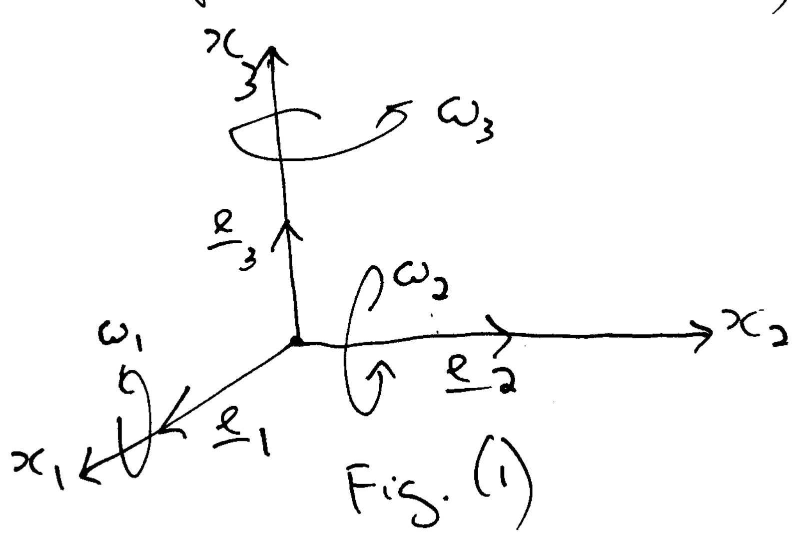

Fig. (1)

a) W.r.t respect to Fig (1):

1) ω_2 rotates $\underline{e}_1$ towards $-\underline{e}_3$

2) ω_3 rotates $\underline{e}_1$ toward $\underline{e}_2$.

It follows that:

$$\frac{d\underline{e}_1}{dt} = \omega_3 \underline{e}_2 - \omega_2 \underline{e}_3 \quad -(8)$$

$$\frac{d\underline{e}_2}{dt} = -\omega_3 \underline{e}_1 + \omega_1 \underline{e}_3 \quad -(9)$$

$$\frac{d\underline{e}_3}{dt} = \omega_2 \underline{e}_1 - \omega_1 \underline{e}_2 \quad -(10)$$

i.e.

$$\dot{\underline{e}}_i = \underline{\omega} \times \underline{e}_i \quad -(11)$$

where

$$i = 1, 2, 3$$

and

$$\underline{\omega} = \omega_1 \underline{e}_1 + \omega_2 \underline{e}_2 + \omega_3 \underline{e}_3 \quad -(12)$$

It follows that:

$$\left(\frac{dr}{dt}\right)_{xyz} = \left(\frac{dr}{dt}\right)_{123} + \sum_i \underline{\omega} \times x_i \underline{e}_i \quad -(13)$$

i.e.

$$\left(\frac{dr}{dt}\right)_{xyz} = \left(\frac{dr}{dt} + \underline{\omega} \times \underline{r}\right)_{123} \quad -(14)$$

It also follows that:

$$\underline{\omega} = \omega_x \underline{i} + \omega_y \underline{j} + \omega_z \underline{k} = \omega_1 \underline{e}_1 + \omega_2 \underline{e}_2 + \omega_3 \underline{e}_3 \quad -(15)$$

and

$$\omega_1^2 + \omega_2^2 + \omega_3^2 = \omega_x^2 + \omega_y^2 + \omega_z^2 \quad -(16)$$

Note that in frame (1, 2, 3):

$$\underline{\omega} \times \underline{r} = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ r_1 & r_2 & r_3 \end{vmatrix} \qquad -(17)$$

$$= (\omega_2 r_3 - \omega_3 r_2)\,\underline{e}_1 + (\omega_3 r_1 - \omega_1 r_3)\,\underline{e}_2 + (\omega_1 r_2 - \omega_2 r_1)\,\underline{e}_3$$

where

$$\underline{r} = r_1 \underline{e}_1 + r_2 \underline{e}_2 + r_3 \underline{e}_3 . \qquad -(18)$$

Therefore:

$$\left(\frac{d\underline{r}}{dt}\right)_{XYZ} = \left(\frac{d\underline{r}}{dt} + (\omega_2 r_3 - \omega_3 r_2)\,\underline{e}_1 + (\omega_3 r_1 - \omega_1 r_3)\,\underline{e}_2 + (\omega_1 r_2 - \omega_2 r_1)\,\underline{e}_3\right)_{123} \qquad -(19)$$

Eq. (19) can be expressed as:

$$\begin{bmatrix} V_X \\ V_Y \\ V_Z \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \qquad -(20)$$

Eq. (20) is true for any vector $\underline{F}$, so:

$$\begin{bmatrix} F_X \\ F_Y \\ F_Z \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \qquad -(21)$$

Eq. (21) is a special case of the Cartan covariant derivative:

$$\frac{DV^a}{\partial x^\mu} = \frac{\partial V^a}{\partial x^\mu} + \Omega^a_{\ \mu b} V^b \quad -(22)$$

for $\mu = 0$. So the well known result (21) can be generalized, giving fundamentally important results.

In eq. (22), $\Omega^a_{\ \mu b}$ is the Cartan spin connection.

Considering space components in eq. (22):

$$\left.\begin{array}{l} a = 1, 2, 3 \\ b = 1, 2, 3 \end{array}\right\} \quad -(23)$$

then:

$$\frac{DV^1}{\partial t} = \frac{\partial V^1}{\partial t} + \Omega^1_{\ 01} V^1 + \Omega^1_{\ 02} V^2 + \Omega^1_{\ 03} V^3$$
$$-(24)$$

etc. It follows that:

$$\frac{d}{dt}\begin{bmatrix} F_X \\ F_Y \\ F_Z \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} + \begin{bmatrix} \Omega^1_{\ 01} & \Omega^1_{\ 02} & \Omega^1_{\ 03} \\ \Omega^2_{\ 01} & \Omega^2_{\ 02} & \Omega^3_{\ 02} \\ \Omega^3_{\ 01} & \Omega^3_{\ 02} & \Omega^3_{\ 03} \end{bmatrix}\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \quad -(25)$$

so the spin connection matrix is:

$$\begin{bmatrix} \Omega^1_{\ 01} & \Omega^1_{\ 02} & \Omega^1_{\ 03} \\ \Omega^2_{\ 01} & \Omega^2_{\ 02} & \Omega^3_{\ 02} \\ \Omega^3_{\ 01} & \Omega^3_{\ 02} & \Omega^3_{\ 03} \end{bmatrix} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \quad -(26)$$

In general therefore, for any vector $\underline{F}$:

$$\left(\frac{d\underline{F}}{dt}\right)_{XYZ} = \left(\frac{d\underline{F}}{dt}\right)_{123} + \left(\underline{\omega} \times \underline{F}\right)_{123} \quad -(27)$$

). Consider the torque $\underline{Tq}$, defined by:

$$\underline{Tq} = \left(\frac{d\underline{L}}{dt}\right)_{XYZ} = \left(\frac{d\underline{L}}{dt} + \underline{\omega} \times \underline{L}\right)_{123} \quad -(28)$$

where $\underline{L}$ is the angular momentum. Using:

$$L_i = I_i \omega_i \quad -(29)$$

where I_i are the three principal moments of inertia for

$$i = 1, 2, 3 \quad -(30)$$

the right hand side of eq. (28) becomes Euler's equations for a rigid body in a force field. In frame (1, 2, 3):

$$\begin{aligned} Tq_1 &= I_1 \dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 \\ Tq_2 &= I_2 \dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 \\ Tq_3 &= I_3 \dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 \end{aligned} \quad -(29)$$

Note that the Cartan spin connection for eqns (29) is given by eq. (26).

The torque vector is defined by:

$$\begin{aligned} \underline{Tq} &= Tq_X \underline{i} + Tq_Y \underline{j} + Tq_Z \underline{k} \\ &= Tq_1 \underline{e}_1 + Tq_2 \underline{e}_2 + Tq_3 \underline{e}_3 \end{aligned} \quad -(30)$$

so:

$$Tq_X^2 + Tq_Y^2 + Tq_Z^2 = Tq_1^2 + Tq_2^2 + Tq_3^2 \quad -(31)$$

6) Similarly, the force components in the frame (1, 2, 3) are:

$$F_1 = m\left(\dot{v}_1 + \omega_2 v_3 - \omega_3 v_2\right)$$
$$F_2 = m\left(\dot{v}_2 + \omega_3 v_1 - \omega_1 v_3\right) \quad -(32)$$
$$F_3 = m\left(\dot{v}_3 + \omega_1 v_2 - \omega_2 v_1\right)$$

The spin connection is again given by Eq. (26). The force vector is: $\quad -(33)$

$$\underline{F} = F_x \underline{i} + F_y \underline{j} + F_Z \underline{k} = F_1 \underline{e}_1 + F_2 \underline{e}_2 + F_3 \underline{e}_3$$

so
$$F_X^2 + F_Y^2 + F_Z^2 = F_1^2 + F_2^2 + F_3^2 \quad -(34)$$

Using plane polar coordinates in orbital theory, eq. (34) reduces to:

$$F_r = -\frac{mMG}{r^2} = \ddot{r} - r\dot{\theta}^2 \quad -(35)$$

and
$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \quad -(36)$$

because:
$$\underline{F} = -\frac{mMG}{r^2}\underline{e}_r = \left(\ddot{r} - r\dot{\theta}^2\right)\underline{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underline{e}_\theta \quad -(37)$$

The spin connection is again eq. (26).

It is clear from eqs. (31) and (34) that the application of an additional torque or force in the ~~[illegible]~~ frame will change ω_1, ω_2 and ω_3 of a gyroscope.

) These angular velocities of frame (1, 2, 3) are related
, the Euler angles θ, ϕ and ψ by:

$$\omega_1 = \dot{\phi}_1 + \dot{\theta}_1 + \dot{\psi}_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$

$$\omega_2 = \dot{\phi}_2 + \dot{\theta}_2 + \dot{\psi}_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \quad -(38)$$

$$\omega_3 = \dot{\phi}_3 + \dot{\theta}_3 + \dot{\psi}_3 = \dot{\phi}\cos\theta + \dot{\psi}$$

It is possible to solve the above equations using various models for the torque, but it is easier to use the Lagrangian method with:

$$\mathcal{L} = T(\text{trans}) + T(\text{rot}) - U \quad -(39)$$

If $\underline{r}$ is the coordinate of the centre of mass of the gyroscope ~~the~~ the translational kinetic energy is:

$$T(\text{trans}) = \frac{1}{2} m \dot{\underline{r}} \cdot \dot{\underline{r}} \quad -(40)$$

and the rotational kinetic energy is:

$$T(\text{rot}) = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right) \quad -(41)$$

In general, the potential energy is:

$$U = U(\underline{r}, \phi, \theta, \psi) - (42)$$

and there are four Euler Lagrange equations in the four Lagrange variables $\underline{r}$, ϕ, θ, and ψ. These can be solved simultaneously to give all the required information and are:

$$\underline{\nabla} \mathcal{L} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\underline{r}}} \quad - (43)$$

and
$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad - (44)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad - (45)$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \quad - (46)$$

Eqs. (39) and (40) give:

$$\underline{F} = m\ddot{\underline{r}} = -\underline{\nabla} U \quad - (47)$$

so any force can be described in terms of a given potential U. In general, U is a function of θ, ϕ and φ, so the force is related to θ, ϕ and φ. In general, eqs. (44), (45) and (46) will contain terms in $\partial U / \partial \phi$, $\partial U / \partial \theta$ and $\partial U / \partial \varphi$.