369(4): The Spin Connection of the Spherical Polar System

Recall from note 363(3) that the spin connection for the plane polar system is:

$$\underline{\Omega} = \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix} \quad -(1)$$

so the covariant derivative of velocity is:

$$\frac{D}{Dt}\begin{bmatrix} \dot{r} \\ r\dot{\theta} \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} \dot{r} \\ r\dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix}\begin{bmatrix} \dot{r} \\ r\dot{\theta} \end{bmatrix} \quad -(2)$$

It follows that the acceleration $\underline{a}$ in plane polars is:

$$\underline{a} = \frac{D\underline{v}}{dt} = \frac{D}{dt}\left(\dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta\right) \quad -(3)$$

where

$$\frac{D\dot{r}}{dt} = \ddot{r} - r\dot{\theta}^2 \quad -(4)$$

and

$$\frac{D(r\dot{\theta})}{dt} = \frac{d(r\dot{\theta})}{dt} + \dot{\theta}\dot{r} = r\ddot{\theta} + 2\dot{r}\dot{\theta} \quad -(5)$$

so

$$\underline{a} = \left(\ddot{r} - \omega^2 r\right)\underline{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underline{e}_\theta \quad -(6)$$

and

$$\underline{F} = m\underline{a} \quad -(7)$$

Here:

$$\underline{F}_N = m\ddot{r}\,\underline{e}_r \quad -(8)$$

is the Newtonian acceleration. Note carefully

But if there is no spin connection, the Newtonian acceleration is the only one present.

In the presence of a spin connection there are three more forces:

$$\underline{F}\,(\text{centrifugal}) = -m\omega^2 r\,\underline{e}_r \quad -(9)$$

$$\underline{F}\,(\text{Coriolis}) = 2m\dot{r}\,\dot{\theta}\,\underline{e}_\theta \quad -(10)$$

and

$$\underline{F}_3 = mr\ddot{\theta}\,\underline{e}_\theta \quad -(11)$$

These are all due to the spin connection, and are therefore examples of Cartan geometry. The origin of the spin connection is the movement of the axes of the plane polar system.

In orbital theory:

$$\underline{F} = m\underline{g} = -\underline{\nabla}U \quad -(12)$$

where U is the potential energy between a mass m orbiting a mass M. If U is central then:

$$U = -\frac{mMG}{r} \quad -(13)$$

and

$$\underline{F} = m\underline{g} = -\frac{mMG}{r^2}\,\underline{e}_r \quad -(14)$$

The force is given by eq. (6):

$$\underline{F} = m\underline{g} = (\ddot{r} - \omega^2 r)\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e}_\theta$$
$$= -\frac{mMG}{r^2}\underline{e}_r \quad -(15)$$

It follows that for planar orbits:

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \quad -(16)$$

and we obtain:

$$\underline{F} = -\frac{mMG}{r^2}\underline{e}_r = m(\ddot{r} - \omega^2 r)\underline{e}_r \quad -(17)$$

so

$$\ddot{r} = \omega^2 r - \frac{MG}{r^2} \quad -(18)$$

which is the Leibniz equation.

Spherical Polar Coordinates.

The velocity is:

$$\underline{v} = \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta + r\dot{\phi}\sin\theta\,\underline{e}_\phi \quad -(19)$$

so the acceleration is given by:

$$\frac{D}{Dt}\begin{bmatrix} \dot{r} \\ r\dot{\theta} \\ r\sin\theta\,\dot{\phi} \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} \dot{r} \\ r\dot{\theta} \\ r\sin\theta\,\dot{\phi} \end{bmatrix} + \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} \end{bmatrix}\begin{bmatrix} \dot{r} \\ r\dot{\theta} \\ r\sin\theta\,\dot{\phi} \end{bmatrix} \quad -(20)$$

From "Vector analysis problem solver" for example it is known that:

$$\underline{a} = a_r \underline{e}_r + a_\theta \underline{e}_\theta + a_\phi \underline{e}_\phi \quad -(21)$$

where:

$$a_r = \ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2\theta \quad -(22)$$

$$a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\phi}^2 \sin\theta\cos\theta \quad -(23)$$

$$a_\phi = 2\dot{r}\dot{\phi}\sin\phi + 2r\dot{\theta}\dot{\phi}\cos\theta + r\sin\theta\,\ddot{\phi} \quad -(24)$$

So:

$$\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta = \ddot{r} + \Omega_{11}\dot{r} + \Omega_{12}r\dot{\theta} + \Omega_{13}\dot{\phi}r\sin\theta \quad -(25)$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta = \frac{d}{dt}(r\dot{\theta}) + \Omega_{21}\dot{r} + \Omega_{22}r\dot{\theta} + \Omega_{23}r\sin\theta\,\dot{\phi} \quad -(26)$$

$$2\dot{r}\dot{\phi}\sin\phi + 2r\dot{\theta}\dot{\phi}\cos\theta + r\sin\theta\,\ddot{\phi} = \frac{d}{dt}(r\sin\theta\,\dot{\phi}) + \Omega_{31}\dot{r} + \Omega_{32}r\dot{\theta} + \Omega_{33}r\sin\theta\,\dot{\phi} \quad -(27)$$

So:

$$\frac{D}{dt}\begin{bmatrix}\dot{r}\\ r\dot{\theta}\\ r\sin\theta\,\dot{\phi}\end{bmatrix} = \frac{d}{dt}\begin{bmatrix}\dot{r}\\ r\dot{\theta}\\ r\sin\theta\,\dot{\phi}\end{bmatrix} + \begin{bmatrix}0 & -\dot{\theta} & -\sin\theta\,\dot{\phi}\\ \dot{\theta} & 0 & -\dot{\phi}\cos\theta\\ \sin\theta\,\dot{\phi} & \dot{\phi}\cos\theta & 0\end{bmatrix}\begin{bmatrix}\dot{r}\\ r\dot{\theta}\\ r\sin\theta\,\dot{\phi}\end{bmatrix} \quad -(28)$$

The spin connection is therefore:

$$\Omega = \begin{bmatrix}0 & -\dot{\theta} & -\sin\theta\,\dot{\phi}\\ \dot{\theta} & 0 & -\cos\theta\,\dot{\phi}\\ \dots\end{bmatrix} \quad -(29)$$

Note that in the above calculation:

$$\frac{d}{dt}\left(r\dot{\theta}\right) = r\ddot{\theta} + \dot{r}\dot{\theta} \quad -(30)$$

and

$$\frac{d}{dt}\left(r\sin\theta\,\dot{\phi}\right) = \dot{\phi}\,\frac{d}{dt}\left(r\sin\theta\right) + \ddot{\phi}\,r\sin\theta \quad -(31)$$

$$= \dot{\phi}\left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right) + \ddot{\phi}\,r\sin\theta$$

Eq. (28) is an example of a Cartan spin connection being used in a covariant derivative.