369(6): Reformulation of the Spin Connection

In the spherical polar system:

$$\dot{\underline{e}}_r = \dot{\theta}\,\underline{e}_\theta + \dot{\phi}\sin\theta\,\underline{e}_\phi \quad -(1)$$

$$\dot{\underline{e}}_\theta = -\dot{\theta}\,\underline{e}_r + \dot{\phi}\cos\theta\,\underline{e}_\phi \quad -(2)$$

$$\dot{\underline{e}}_\phi = -\dot{\phi}\sin\theta\,\underline{e}_r - \dot{\phi}\cos\theta\,\underline{e}_\theta \quad -(3)$$

("Vector Analysis Problem Solver"). The velocity in the spherical polar coordinate system is:

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta + r\dot{\phi}\sin\theta\,\underline{e}_\phi. \quad -(4)$$

It follows that:

$$\begin{bmatrix} a_r \\ a_\theta \\ a_\phi \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} \dot{r} \\ r\dot{\theta} \\ r\sin\theta\,\dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\theta} & -\dot{\phi}\sin\theta \\ \dot{\theta} & 0 & -\dot{\phi}\cos\theta \\ \dot{\phi}\sin\theta & \dot{\phi}\cos\theta & 0 \end{bmatrix}\begin{bmatrix} \dot{r} \\ r\dot{\theta} \\ r\sin\theta\,\dot{\phi} \end{bmatrix} \quad -(5)$$

can be reexpressed as:

$$a_r = \frac{d\dot{r}}{dt} - \dot{\underline{e}}_r\cdot\underline{v} = \frac{dv_r}{dt} - \dot{\underline{e}}_r\cdot\underline{v} \quad -(6)$$

$$a_\theta = \frac{d}{dt}\left(r\dot{\theta}\right) - \dot{\underline{e}}_\theta\cdot\underline{v} = \frac{dv_\theta}{dt} - \dot{\underline{e}}_\theta\cdot\underline{v} \quad -(7)$$

$$a_\phi = \frac{d}{dt}\left(r\sin\theta\,\dot{\phi}\right) - \dot{\underline{e}}_\phi\cdot\underline{v} = \frac{dv_\phi}{dt} - \dot{\underline{e}}_\phi\cdot\underline{v} \quad -(8)$$

In the plane polar system:

$$\dot{\underline{e}}_r = \dot{\theta}\,\underline{e}_\theta \quad -(9)$$

$$\dot{\underline{e}}_\theta = -\dot{\theta}\,\underline{e}_r \quad -(10)$$

and
$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta. \quad -(11)$$

It follows that:

$$\begin{bmatrix} a_r \\ a_\theta \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} \dot{r} \\ r\dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix}\begin{bmatrix} \dot{r} \\ r\dot{\theta} \end{bmatrix} \quad -(12)$$

can be re expressed as:

$$a_r = \frac{d\dot{r}}{dt} - \dot{\underline{e}}_r \cdot \underline{v} = \frac{dv_r}{dt} - \dot{\underline{e}}_r \cdot \underline{v} \quad -(13)$$

and

$$a_\theta = \frac{d}{dt}\left(r\dot{\theta}\right) - \dot{\underline{e}}_\theta \cdot \underline{v} = \frac{dv_\theta}{dt} - \dot{\underline{e}}_\theta \cdot \underline{v} \quad -(14)$$

Expression such as (6) to (8) and (13), (14) are true for any coordinate system

---