69(7): The Complete Solution of Spherical Orbit Theory in a Central Force Field.

In this case, using spherical polar coordinates:

$$\underline{F} = -\underline{\nabla} U = m\left(a_r \underline{e}_r + a_\theta \underline{e}_\theta + a_\phi \underline{e}_\phi\right) \quad -(1)$$

where

$$U = -\frac{mMG}{r} \quad -(2)$$

So

$$\underline{F} = -\frac{mMG}{r^2}\underline{e}_r = m\left(\frac{d\underline{v}}{dt} + \underline{\omega} \times \underline{v}\right) \quad -(3)$$

where $\underline{\omega}$ is the angular velocity in spherical polar coordinates, and $\underline{v}$ is the linear velocity.

Therefore:

$$\underline{F} = m\left(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2\theta\right)\underline{e}_r = -\frac{mMG}{r^2}\underline{e}_r \quad -(4)$$

i.e.

$$\ddot{r} = r\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) = -\frac{MG}{r^2} \quad -(5)$$

which is the Leibnitz equation for a three dimensional orbit.

For the central force field (3):

$$a_\theta \underline{e}_\theta + a_\phi \underline{e}_\phi = \underline{0} \quad -(6)$$

so

$$a_\theta^2 + a_\phi^2 = 0 \quad -(7)$$

where:

$$a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\,\dot{\phi}^2 \quad - (8)$$

and

$$a_\phi = 2\dot{\phi}\dot{r}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta + r\ddot{\phi}\sin\theta \quad - (9)$$

The Lagrangian is :

$$\mathcal{L} = \frac{1}{2}mv^2 - U \quad - (10)$$

$$= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) + \frac{mMG}{r}$$

and the Euler Lagrange equations are:

$$\frac{\partial\mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} \quad - (11)$$

$$\frac{\partial\mathcal{L}}{\partial\theta} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} \quad - (12)$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} \quad - (13)$$

From eqs. (10) and (11):

$$\ddot{r} = r\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) - \frac{MG}{r^2} \quad - (14)$$

which is eq. (5), Q.E.D.

From eqs. (10) and (13):

$$L_\phi = mr^2\dot{\phi}\sin^2\theta = \text{constant} \quad - (15)$$

i.e

$$\frac{dL_\phi}{dt} = 0 \quad - (16)$$

) Here L_ϕ is a constant of motion, a constant angular momentum.

From eqs. (10) and (16):

$$mr^2 \sin\theta \cos\theta \, \dot{\phi}^2 = \frac{d}{dt}\left(mr^2\dot{\theta}\right) \quad -(17)$$

in which: $r = r(t)$, $-(18)$

so

$$r \sin\theta \cos\theta \, \dot{\phi}^2 = r\ddot{\theta} + 2\dot{\theta}\dot{r} \quad -(19)$$

So the complete set of equations for all spherical orbits in a central force field is:

$$\ddot{r} = r\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) - \frac{mG}{r^2} \quad -(20)$$

$$r \sin\theta \cos\theta \, \dot{\phi}^2 = r\ddot{\theta} + 2\dot{\theta}\dot{r} \quad -(21)$$

$$L_\phi = mr^2 \dot{\phi} \sin^2\theta = \text{constant} \quad -(22)$$

and

$$\left(2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\,\dot{\phi}^2\right)^2 + \left(2\dot{\phi}\dot{r}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta + r\ddot{\phi}\sin\theta\right)^2 = 0 \quad -(23)$$

From eqs. (19) and (23):

$$2\dot{\phi}\dot{r}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta + r\ddot{\phi}\sin\theta = 0 \quad -(24)$$

i.e. $\quad a_\theta = 0, \quad a_\phi = 0 \quad -(25)$

4) Multiply eq. (24) by $\sin\theta$, and use eq. (22) to find:

$$\frac{2\dot{r}L_\phi}{mr^2} + 2r\dot{\phi}\dot{\theta}\cos\theta\sin\theta + r\ddot{\phi}\sin^2\theta = 0 \quad -(26)$$

So eqs (20) to (23) reduce to three simultaneous differential equations in three unknowns: $r(t)$, $\theta(t)$ and $\phi(t)$. These trajectories completely define the orbit. The set of equations is:

$$\ddot{r} = r\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) - \frac{MG}{r^2} \quad -(27)$$

$$r\sin\theta\cos\theta\,\dot{\phi}^2 = r\ddot{\theta} + 2\dot{\theta}\dot{r} \quad -(28)$$

and

$$\frac{2\dot{r}L_\phi}{mr^2} + 2r\dot{\phi}\dot{\theta}\cos\theta\sin\theta + r\ddot{\phi}\sin^2\theta = 0 \quad -(29)$$

and can be solved for $r(t)$, $\theta(t)$ and $\phi(t)$. They can also be solved for $\frac{dr}{dt}$, $\frac{d\theta}{dt}$ and $\frac{d\phi}{dt}$. Therefore the following orbital quantities can be found:

$$\frac{d\theta}{dr} = \frac{dr}{dt}\frac{dt}{d\theta}, \quad -(30)$$

$$\frac{d\phi}{dr} = \frac{dr}{dt}\frac{dt}{d\phi}, \quad -(31)$$

$$\frac{d\theta}{d\phi} = \frac{d\theta}{dt}\frac{dt}{d\phi} \quad -(32)$$

) so the orbits $r(\theta)$ and $r(\phi)$ can be found by integration. The function $\theta(\phi)$ can also be found by integration.

Eqs. (27) to (29) can be solved using the Maxima routine used in UFT 368.

In general there is no reason why an orbit should be planar, however, planar orbit theory is recovered from eqs. (27) to (29) using:

$$\dot{\phi} = 0, \quad \phi = \frac{\pi}{2} \quad -(33)$$

So eqs. (27) to (29) reduce to:

$$\ddot{r} = r\dot{\theta}^2 - \frac{MG}{r^2} \quad -(34)$$

$$r\ddot{\theta} + 2\dot{\theta}\dot{r} = 0 \quad -(35)$$

and

$$L_\phi = 0. \quad -(36)$$

These are the equations of planar orbits used in many UFT papers.

Note carefully that eqs. (34) and (35) can be solved using the Maxima code, because they are two differential equations in two unknowns, r and θ. This solution gives $r(t)$ and $\theta(t)$, so solutions can also be found for dr/dt, $d\theta/dt$ and $dr/d\theta$. The orbit $r(\theta)$ can then be found by integration. It is a conic section on this classical level.