69(8): General Theory of a Gyroscope in an External Force Field.

In general the Lagrangian is:

$$\mathcal{L} = \frac{1}{2} m \underline{\dot{r}} \cdot \underline{\dot{r}} + \frac{1}{2}\left(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2\right) - U \quad -(1)$$

The potential energy in general is defined by:

$$\int_1^2 \underline{F} \cdot d\underline{r} = U_1 - U_2 \quad -(2)$$

in terms of the work done by the force $\underline{F}$ to transport the centre of mass of the gyro from point 1 to point 2. A solution of eq. (2) is:

$$\underline{F} = -\underline{\nabla} U \quad -(3)$$

corresponding to the external torque:

$$\underline{Tq} = \underline{r} \times \underline{F} \quad -(4)$$

Note that U is the same in frames $(1, 2, 3)$ and (X, Y, Z) because it is a scalar. In general:

$$T(\text{rot}) = \frac{1}{2}\left(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2\right) \quad -(5)$$

where

$$\omega_1 = \dot{\phi}_1 + \dot{\theta}_1 + \dot{\psi}_1 = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi \quad -(6)$$

$$\omega_2 = \dot{\phi}_2 + \dot{\theta}_2 + \dot{\psi}_2 = \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi \quad -(7)$$

$$\omega_3 = \dot{\phi}_3 + \dot{\theta}_3 + \dot{\psi}_3 = \dot{\phi} \cos\theta + \dot{\psi} \quad -(8)$$

2) For a symmetric top:

$$T(\text{rot}) = \frac{1}{2} I_{12} \left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2 \right) + \frac{1}{2} I_3 \left(\dot{\phi} \cos\theta + \dot{\psi} \right)^2 \quad -(9)$$

and the general lagrangian is:

$$\mathcal{L} = \frac{1}{2} m \underline{\dot{r}} \cdot \underline{\dot{r}} + T(\text{rot}) - (U_1 - U_2) \quad -(10)$$

$$= \frac{1}{2} m \underline{\dot{r}} \cdot \underline{\dot{r}} + T(\text{rot}) - \int_1^2 \underline{F} \cdot \underline{dr} \quad -(11)$$

in which any kind of external force can be used.

For the symmetric top with one point fixed:

$$\mathcal{L} = T(\text{rot}) - mgh\cos\theta \quad -(12)$$

and

$$\underline{\dot{r}} = \underline{0}, \quad -(13)$$

meaning that the point of the gyro does not translate.

Eq. (12) generalizes to:

$$\mathcal{L} = \frac{1}{2} m \underline{\dot{r}} \cdot \underline{\dot{r}} + T(\text{rot}) - mgh\cos\theta - \int_1^2 \underline{F} \cdot \underline{dr} \quad -(14)$$

in which

$$\underline{\dot{r}} \neq \underline{0}. \quad -(15)$$

Writing

$$U := U_1 - U_2 \quad -(16)$$

then

$$\mathcal{L} = \frac{1}{2} m \underline{\dot{r}} \cdot \underline{\dot{r}} + T(\text{rot}) - mgh\cos\theta - U \quad -(15)$$

8) If it is assumed that U is a function only of r then:

$$\mathcal{L} = \frac{1}{2} m \dot{\underline{r}} \cdot \dot{\underline{r}} + T(\text{rot}) - mgl\cos\theta - U(r) \quad -(16)$$

and there are four Euler Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(17)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad -(18)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad -(19)$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \quad -(20)$$

These give:

$$\dot{\phi} = \frac{L_\phi - L_\varphi \cos\theta}{I_{12}\sin^2\theta} \quad -(21)$$

$$\dot{\varphi} = \frac{1}{I_3}\left(L_\varphi - I_3 \dot{\phi}\cos\theta\right) \quad -(22)$$

$$\ddot{\theta} = \frac{\sin\theta}{I_{12}}\left(\dot{\phi}^2\cos\theta\left(I_{12} - I_3\right) - I_3\dot{\phi}\dot{\varphi} + mgl\right) \quad -(23)$$

and

$$m\frac{d\ddot{\underline{r}}}{dt} = -\underline{\nabla} U(r) \quad -(24)$$

If it is assumed that:

$$U = U(r, \theta) \quad -(25)$$

then eq. (23) changes to:

$$\ddot{\theta} = \frac{\sin\theta}{I_{12}}\left(\dot{\phi}^2 \cos\theta\,(I_{12} - I_3) - I_3\,\dot{\phi}\,\dot{\psi} + mgh\right) + \frac{1}{I_{12}}\frac{\partial U(r,\theta)}{\partial \theta} \quad -(26)$$

In the theory of Milankovitch cycles:

$$\mathcal{L} = \frac{1}{2} m\, \dot{\underline{r}} \cdot \dot{\underline{r}} + T(\text{rot}) - U. \quad -(27)$$

The three dimensional version of eq. (17) is:

$$\underline{\nabla} \mathcal{L} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\underline{r}}} \quad -(28)$$

so:

$$\dot{\phi} = \frac{L_\phi - L_\psi \cos\theta}{I_{12} \sin^2\theta} \quad -(29)$$

$$\dot{\psi} = \frac{1}{I_3}\left(L_\psi - I_3\,\dot{\phi}\cos\theta\right) \quad -(30)$$

$$\ddot{\theta} = \frac{\sin\theta}{I_{12}}\left(\dot{\phi}^2 \cos\theta\,(I_{12} - I_3) - I_3\,\dot{\phi}\,\dot{\psi}\right) \quad -(31)$$

and

$$m \frac{d^2 \underline{r}}{dt^2} = -\underline{\nabla} U \quad -(32)$$

where

$$U = -\frac{mMG}{r} \quad -(33)$$

For a central potential the Mlaurcovit cycle re found by solving Eqs. (29) to (31) for ϕ, $\dot{\phi}$, ϕ, $\dot{\theta}$ and $\dot{\psi}$.

In plane polar coordinates (r, θ_1), eqs. (32) and (33) give:

$$\ddot{r} - r\dot{\theta}_1^2 = -\frac{MG}{r^2} \quad -(34)$$

$$L_{\theta_1} = mr^2\dot{\theta}_1 = \text{constant} \quad -(35)$$

and

$$r\ddot{\theta}_1 + 2\dot{\theta}_1\dot{r} = 0 \quad -(36)$$

Eqs. (34) to (36) can be solved to give r, θ_1, $\dot{r}$ and $\dot{\theta}_1$ and $dr/d\theta_1$.

If there is no interaction between rotation and translation Eqs. (29) to (31) and Eqs. (34) to (36) are independent.

However, if there is a functional relation between θ_1 and the Euler angles (θ, ϕ, ψ), there is interaction between rotation and translation:

$$\theta_1 = \theta_1(\theta, \phi, \psi) \quad -(37)$$

and the nutations and precessions are affected.