369(9): General Theory of Milankovitch Cycles

Fig (1)

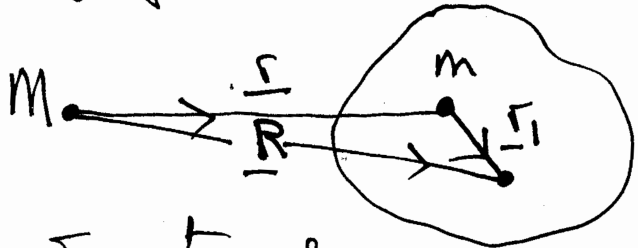

Fig (1)

Consider any asymmetric top of mass m orbiting the sun of mass $\underline{M}$. The distance between the centre of mass of the object and the sun is $\underline{r}$. The distance of any point in the object from its centre of mass is $\underline{r}_1$. Therefore the distance between the sun and this point is

$$\underline{R} = \underline{r} + \underline{r}_1 \quad -(1)$$

The lagrangian of relevance is:

$$\mathcal{L} = \frac{1}{2} m \left(\dot{\underline{r}} + \dot{\underline{r}}_1 \right) \cdot \left(\dot{\underline{r}} + \dot{\underline{r}}_1 \right) - U \quad -(2)$$

where

$$U = -\frac{mMG}{r} \quad -(3)$$

The position of the centre of mass is denoted:

$$\underline{r} = X\,\underline{i} + Y\,\underline{j} + Z\,\underline{k} \quad -(4)$$

in the laboratory frame.

The position of $\underline{r}_1$ is:

$$\underline{r}_1 = r_{11}\,\underline{e}_1 + r_{12}\,\underline{e}_2 + r_{13}\,\underline{e}_3 \quad -(5)$$

in the frame (1, 2, 3) of the principal moments of inertia

2) of the asymmetric top.

The transformation from $(1,2,3)$ to (X,Y,Z) is given by:

$$\begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \\ \underline{e}_3 \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{i} \\ \underline{j} \\ \underline{k} \end{bmatrix} \quad -(6)$$

where θ, ϕ and ψ are the Euler angles defined in Marion and Thornton for example. So:

$$\begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \\ \underline{e}_3 \end{bmatrix} = \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} \underline{i} \\ \underline{j} \\ \underline{k} \end{bmatrix} \quad -(7)$$

where:

$$\begin{aligned}
A &= \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi \\
B &= -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi \\
C &= \sin\theta\cos\phi \\
D &= \cos\psi\cos\phi + \cos\theta\cos\phi\sin\psi \\
E &= -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi \\
F &= -\sin\theta\cos\phi \\
G &= \sin\psi\cos\theta \\
H &= \cos\psi\sin\theta \\
I &= \cos\theta
\end{aligned} \quad -(8)$$

It follows that:

$$\underline{r}_1 = r_{11}(A\underline{i} + B\underline{j} + C\underline{k}) + r_{12}(D\underline{i} + E\underline{j} + F\underline{k}) + r_{13}(G\underline{i} + H\underline{j} + I\underline{k})$$

$$= (r_{11}A + r_{12}D + r_{13}G)\underline{i} + (r_{11}B + r_{12}E + r_{13}H)\underline{j} + (r_{11}C + r_{12}F + r_{13}I)\underline{k} \quad -(9)$$

$$= X_1\underline{i} + Y_1\underline{j} + Z_1\underline{k}$$

where:

$$X_1 = r_{11}A + r_{12}D + r_{13}G$$
$$Y_1 = r_{11}B + r_{12}E + r_{13}H \quad -(10)$$
$$Z_1 = r_{11}C + r_{12}F + r_{13}I$$

The Lagrangian is therefore:

$$\mathcal{L} = \frac{1}{2}m\left((\dot{X}+\dot{X}_1)^2 + (\dot{Y}+\dot{Y}_1)^2 + (\dot{Z}+\dot{Z}_1)^2\right) + \frac{mMG}{r} \quad -(11)$$

If the orbit is considered to be planar, then:

$$X = r\cos\theta_1 \quad -(12)$$
$$Y = r\sin\theta_1 \quad -(13)$$
$$Z = 0 \quad -(14)$$

) The system is governed by five Euler Lagrange
equations in five Lagrange variables, r, θ_1, θ, ϕ and ψ:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) \quad -(15)$$

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) \quad -(16)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) \quad -(17)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) \quad -(18)$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}}\right) \quad -(19)$$

These five equations must be solved simultaneously.

In eq. (11):

$$\dot{X} = \dot{r}\cos\theta_1 - r\dot{\theta}_1\sin\theta_1 \quad -(20)$$

$$\dot{Y} = \dot{r}\sin\theta_1 + r\dot{\theta}_1\cos\theta_1 \quad -(21)$$

$$\dot{Z} = 0 \quad -(22)$$

The distances r_{11}, r_{12} and r_{13} are constants
related to the principal moments of inertia I_1,
I_2 and I_3 of the asymmetric top.

) For the purposes of computing these can be used asput parameters. Therefore:

$$\dot{X}_1 = r_{11}\frac{dA}{dt} + r_{12}\frac{dD}{dt} + r_{13}\frac{dG}{dt} \quad -(23)$$

$$\dot{Y}_1 = r_{11}\frac{dB}{dt} + r_{12}\frac{dE}{dt} + r_{13}\frac{dH}{dt} \quad -(24)$$

$$\dot{Z}_1 = r_{11}\frac{dC}{dt} + r_{12}\frac{dF}{dt} + r_{13}\frac{dI}{dt} \quad -(25)$$

and can be evaluated by computer algebra to eliminate human error.

The Lagrangian is therefore:

$$\mathcal{L} = \frac{1}{2}m\left(\left(\dot{r}\cos\theta_1 - r\dot{\theta}_1\sin\theta_1 + \dot{X}_1\right)^2 + \left(\dot{r}\sin\theta_1 + r\dot{\theta}_1\cos\theta_1 + \dot{Y}_1\right)^2 + \dot{Z}_1^2\right) + \frac{mMG}{r} \quad -(26)$$

It is seen that the Milankovitch cycles are the result of intricate dynamics - the motion of the rotating asymmetric top superimposed on its orbital motion. All the motions are inter-related.