369(2): Gyro Attached to a Stand

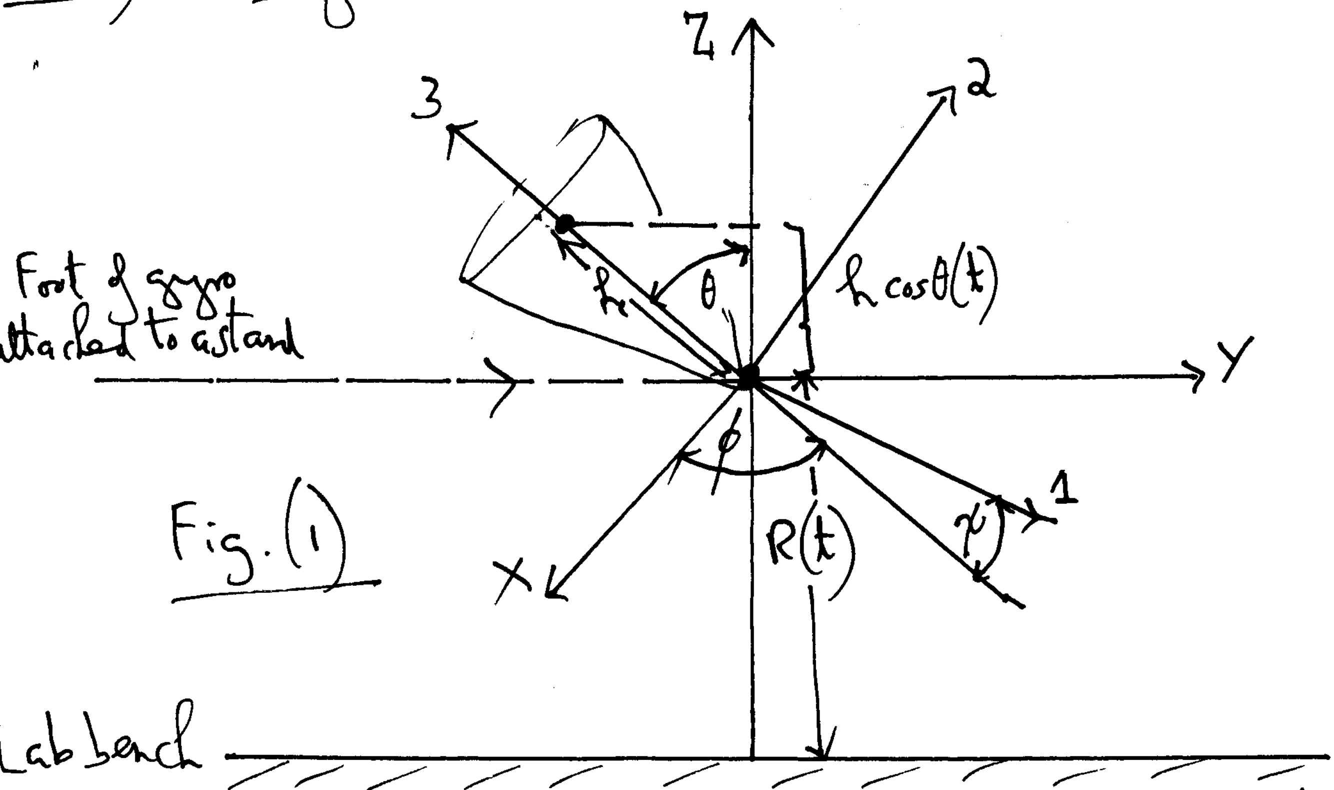

Fig. (1)

The diagram describes a gyroscope with its foot attached to a laboratory stand in the Z axis. The origins of (1, 2, 3) and (X, Y, Z) coincide with the foot of the gyroscope. The centre of mass of the gyroscope is a constant distance h from the origin along the 3 axis of the principal moment of inertia frame (1, 2, 3). The distance of the point of the gyroscope above the laboratory bench is R(t). If the point of the gyroscope is attached to the stand with a slow friction collar, so R can vary with time. If the point is attached firmly to the stand then R is a constant. The Euler angles are as defined in the diagram.

The Lagrangian of the system is:

2)

$$\mathcal{L} = T - U \quad \text{---} (1)$$

where

$$U = mg\left(h\cos\theta(t) + R(t)\right) - (2)$$

Here m is the mass of the gyro and g the acceleration due to gravity

So $$\mathcal{L} = T - mg\left(h\cos\theta(t) + R(t)\right). - (3)$$

Define: $$R_1(t) = h\cos(t) + R(t). - (4)$$

The total kinetic energy is:

$$T = T(\text{translation}) + T(\text{rotation}) - (5)$$

where $$T(\text{translation}) = \frac{1}{2} m \dot{R}_1^2 - (6)$$

and $$T(\text{rotation}) = \frac{1}{2} I_{12}\left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2} I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2$$

$$- (7)$$

As in Note 3(a)(1) the Euler Lagrange equations wrt Lagrange variables R_1, θ, ϕ and ψ give four simultaneous differential equations:

$$\dot{\phi} = \frac{L_\phi - L_\psi\cos\theta}{I_{12}\sin^2\theta} - (8)$$

$$\dot{\psi} = \frac{1}{I_3}\left(L_\psi - I_3\dot{\phi}\cos\theta\right) - (9)$$

$$\ddot{\theta} = \frac{\sin\theta}{I_{12}}\left(\dot{\phi}^2\cos\theta\left(I_{12}-I_3\right) - I_3\dot{\phi}\dot{\psi} + mgh\right) \quad -(10)$$

There is also an equation:

$$\frac{\partial \mathcal{L}}{\partial R_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{R}_1} \quad -(11)$$

where the total lagrangian is:

$$\mathcal{L} = \frac{1}{2}m\dot{R}_1^2 + \frac{1}{2}I_{12}\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2 - mgR_1 \quad -(12)$$

where:

$$R_1 = h\cos\theta + R \quad -(13)$$

From eqs. (11) and (12):

$$g = -\ddot{R}_1 \quad -(14)$$

$$= -\left(\ddot{R} + h\frac{d^2\cos\theta(t)}{dt^2}\right)$$

$$= \ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta - \ddot{R}$$

So

$$\boxed{\ddot{R}(t) = \ddot{\theta}(t)\sin\theta(t) + \dot{\theta}^2(t)\cos\theta(t) - g \quad -(14)}$$

Eqs. (8) to (10) and (14) must be solved simultaneously for the trajectories of R, θ, ϕ and ψ.

) This gives the general solution of the problem, and the general solution answers the problem of whether the gyroscope can be elevated by its own spin.

If the point of the gyroscope is fixed to the stand, then:

$$\ddot{R}(t) = 0 \quad -(15)$$

so

$$g = \ddot{\theta}(t)\sin\theta(t) + \dot{\theta}^2(t)\cos\theta(t)$$

$$-(16)$$

Eqs. (15) and (16) are constraint equations, and the problem reduces to solving eqs. (8) to (10) and (16) simultaneously.

The gyroscope would appear to be weightless if the force due to gravitation:

$$F = mg \quad -(17)$$

is counterbalanced by a force in the positive Z axis:

$$F = m\left(\ddot{\theta}(t)\sin\theta(t) + \dot{\theta}^2(t)\cos\theta(t)\right). \quad -(18)$$

The acceleration due to gravity is:

$$g = -\frac{MG}{r_e^2} \quad -(19)$$

where r_e is the radius of the earth, M is the mass of earth, and G is the Newton constant. So the condition of weightlessness is:

$$-\frac{MG}{r_e^2} + \ddot{\theta}(t)\sin\theta(t) + \dot{\theta}^2(t)\cos\theta(t)$$

$$0 \quad -(20)$$

) In the Lashwante experiment:

$$\theta = \frac{\pi}{2} \quad -(21)$$

itially, so

$$\ddot{\theta} = \frac{mG}{r_e^2} \quad -(22)$$

nitially.