69(3): General Theory of the Earth's Precession (Milankovitch Cycles)

First consider the lagrangian for the earth considered as a free symmetric top. This is an idealization because the earth is always influenced by the gravitational fields of the sun, moon and planets.

The lagrangian is:

$$\mathcal{L} = \frac{1}{2}\left(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2\right) \quad -(1)$$

where

$$I_{12} = I_1 = I_2 \quad -(2)$$

and I_3 are the principal moments of inertia of the earth, and ω_1, ω_2 and ω_3 are angular velocity components in the principal moment of inertia frame (1, 2, 3). Here:

$$\omega_1 = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi \quad -(3)$$

$$\omega_2 = \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi \quad -(4)$$

$$\omega_3 = \dot{\phi} \cos\theta + \dot{\psi} \quad -(5)$$

The Euler Lagrange equations are:

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad -(6)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad -(7)$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \quad -(8)$$

The motion of the free earth is found by solving eqs. (6) to (8) simultaneously for the trajectories $\theta(t)$, $\phi(t)$ and $\psi(t)$. These are the trajectories of the earth in a very weak gravitational field.

When the earth of mass m is in the gravitational field of an object such as the sun of mass $\underline{M}$, the geometry is as in Fig (1).

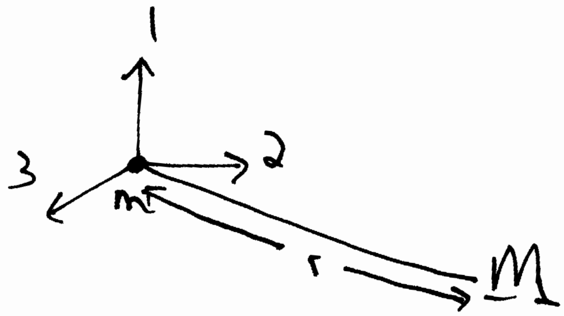

Fig (1)

and the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2} m \underline{\dot{r}} \cdot \underline{\dot{r}} + \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right) - U \quad -(9)$$

where

$$U = -\frac{mMG}{r} \quad -(10)$$

For self consistency the vector $\underline{\dot{r}}$ and the scalar must be transformed into frame (1, 2, 3). In Cartesian coordinates:

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad -(11)$$

It is convenient to then transform into spherical polar coordinates:

$$\begin{aligned} X &= r\sin\theta\cos\phi \\ Y &= r\sin\theta\sin\phi \\ Z &= r\cos\theta \end{aligned} \qquad -(12)$$

so:

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = r \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix} \qquad -(13)$$

where

$$r = r(lab) \qquad -(14)$$

meaning that r is defined in the lab frame. The angles θ and ϕ of the spherical polar coordinate system are the same as the Euler angle θ and ϕ. Therefore in the $(1, 2, 3)$ frame:

$$\begin{aligned} r_1 &= r(lab)\, f_1(\theta, \phi, \psi) \\ r_2 &= r(lab)\, f_2(\theta, \phi, \psi) \\ r_3 &= r(lab)\, f_3(\theta, \phi, \psi) \end{aligned} \qquad -(15)$$

and

$$\underline{r}(1, 2, 3) = r_1\,\underline{e}_1 + r_2\,\underline{e}_2 + r_3\,\underline{e}_3 \qquad -(16)$$

so

$$r(1, 2, 3) = \left(f_1^2 + f_2^2 + f_3^2\right)^{1/2} r(lab) \qquad -(17)$$

Therefore the potential energy in frame (1, 2, 3) is

$$U = -\frac{mMG}{r(1,2,3)} \left(f_1^2 + f_2^2 + f_3^2 \right)^{1/2} \quad -(18)$$

There is an additional Euler Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial r(1,2,3)} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}(1,2,3)} \quad -(19)$$

The precession of the Earth in the sun's gravitational field are therefore described by the simultaneous solution of eqs. (6), (7), (8) and (19) to give the trajectories $r(t)$, $\theta(t)$, $\phi(t)$ and $\varphi(t)$.

This is the general theory of the precession of the equinox, Milankovich cycles, and similar.