365(2): Spin Connection for the General Precessing Ellipse.

In this case the two equations to be solved are:

$$\left(1+\Omega^1{}_{01}\right)\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{m^2MG}{L^2} \quad -(1)$$

for the general precessing ellipse:

$$\frac{1}{r}=\frac{1}{\alpha}\left(1+\epsilon\cos\left(\theta f(\theta)\right)\right) \quad -(2)$$

where $f(\theta)$ is to be determined. For small precessions:

$$\frac{m^2MG}{L^2}=\frac{1}{\alpha} \quad -(3)$$

So:

$$\left(1+\Omega^1{}_{01}\right)\frac{d^2}{d\theta^2}\left(\frac{1}{\alpha}\left(1+\epsilon\cos\left(\theta f(\theta)\right)\right)\right)+\frac{1}{\alpha}\left(1+\epsilon\cos\left(\theta f(\theta)\right)\right)=\frac{1}{\alpha} \quad -(4)$$

i.e.

$$\left(1+\Omega^1{}_{01}\right)\frac{d^2}{d\theta^2}\cos\left(\theta f(\theta)\right)+\cos\left(\theta f(\theta)\right)=0 \quad -(5)$$

If $$f(\theta)=1 \quad -(6)$$

then $$\Omega^1{}_{01}=0 \quad -(7)$$

otherwise $\Omega^1{}_{01}$ can be expressed in terms of $f(\theta)$ by solving eq. (5).

2) Let $$y = \cos(\theta f(\theta)) \quad -(8)$$

then $$\frac{d^2y}{d\theta^2} = \frac{-y}{(1+\Omega'_{01})} \quad -(9)$$

so $$y = \exp\left(\frac{i\theta}{(1+\Omega'_{01})^{1/2}}\right) \quad -(10)$$

$$= \cos\left(\frac{\theta}{(1+\Omega'_{01})^{1/2}}\right) + i\sin\left(\frac{\theta}{(1+\Omega'_{01})^{1/2}}\right)$$

So $$\text{Real } y = \cos\left(\frac{\theta}{(1+\Omega'_{01})^{1/2}}\right) = \cos(\theta f(\theta)). \quad -(11)$$

It follows that:

$$f(\theta) = \frac{1}{(1+\Omega'_{01})^{1/2}} \quad -(12)$$

and $$\Omega'_{01} = \frac{1}{f^2(\theta)} - 1 \quad -(13)$$

where $$f(\theta) = \frac{1}{\theta}\cos^{-1}\left(\frac{\alpha}{r} - 1\right) \quad -(14)$$

Under these conditions the inverse square law produces the general precessing ellipse (2)

) Therefore the general precessing ellipse or conic function is:

$$\boxed{r = \frac{\alpha}{1 + \epsilon \cos\left(\frac{\theta}{\left(1 + \Omega'_{01}\right)^{1/2}}\right)}} \quad -(15)$$

where

$$\Omega'_{01} = \frac{\partial R}{\partial r} \quad -(16)$$

Eq. (15) is produced by the inverse square law used in the fluid dynamic Binet equation (1).