362(5): Convective Derivative of the General Vector.

This is defined as:

$$\frac{D\underline{V}}{Dt} = \frac{\partial \underline{V}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{V} \quad -(1)$$

where $\underline{v}(t, r, \theta)$ is the velocity field and where:

$$\underline{V} = \underline{V}(t, r(t), \theta(t)) \quad -(2)$$

Therefore, in plane polar coordinates:

$$\frac{D\underline{V}}{Dt} = \frac{\partial \underline{V}}{\partial t} + \left(v_r \frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial \theta}\right)\left(V_r \underline{e}_r + V_\theta \underline{e}_\theta\right)$$

$$= \frac{\partial \underline{V}}{\partial t} + v_r \frac{\partial}{\partial r}\left(V_r \underline{e}_r\right) + \frac{v_\theta}{r}\frac{\partial}{\partial \theta}\left(V_r \underline{e}_r\right)$$

$$+ v_r \frac{\partial}{\partial r}\left(V_\theta \underline{e}_\theta\right) + \frac{v_\theta}{r}\frac{\partial}{\partial \theta}\left(V_\theta \underline{e}_\theta\right)$$

$$= \frac{\partial \underline{V}}{\partial t} + v_r\left(\frac{\partial V_r}{\partial r}\underline{e}_r + V_r \frac{\partial \underline{e}_r}{\partial r}\right) + \frac{v_\theta}{r}\left(\frac{\partial V_r}{\partial \theta}\underline{e}_r + V_r\frac{\partial \underline{e}_r}{\partial \theta}\right) \quad -(3)$$

$$+ v_r\left(\frac{\partial V_\theta}{\partial r}\underline{e}_\theta + V_\theta\frac{\partial \underline{e}_\theta}{\partial r}\right) + \frac{v_\theta}{r}\left(\frac{\partial V_\theta}{\partial \theta}\underline{e}_\theta + V_\theta\frac{\partial \underline{e}_\theta}{\partial \theta}\right)$$

Now use the plane polar coordinate results:

$$\frac{\partial \underline{e}_r}{\partial r} = \underline{0}, \quad \frac{\partial \underline{e}_\theta}{\partial r} = \underline{0}, \quad -(4)$$

$$\frac{\partial \underline{e}_r}{\partial \theta} = \underline{e}_\theta, \quad \frac{\partial \underline{e}_\theta}{\partial \theta} = -\underline{e}_r.$$

2) to obtain:

$$\frac{D\underline{V}}{Dt} = \frac{\partial \underline{V}}{\partial t} + \left(V_r \frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta}{r} V_\theta \right) \underline{e}_r \quad -(5)$$

$$+ \left(V_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_\theta}{r} V_r \right) \underline{e}_\theta$$

where

$$\frac{V_\theta}{r} = \dot{\theta} = \frac{d\theta}{dt} = \omega \quad -(6)$$

In component format:

$$\frac{D}{Dt}\begin{bmatrix} V_r \\ V_\theta \end{bmatrix} = \frac{\partial}{\partial t}\begin{bmatrix} V_r \\ V_\theta \end{bmatrix} + \begin{bmatrix} \frac{\partial V_r}{\partial r} & \frac{1}{r}\frac{\partial V_r}{\partial \theta} \\ \frac{\partial V_\theta}{\partial r} & \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} \end{bmatrix} \begin{bmatrix} V_r \\ V_\theta \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix} \begin{bmatrix} V_r \\ V_\theta \end{bmatrix} \quad -(7)$$

where

$$\underline{V} = V_r(t, r(t), \theta(t))\,\underline{e}_r + V_\theta(t, r(t), \theta(t))\,\underline{e}_\theta \quad -(8)$$

is a vector field of fluid dynamics.

In classical dynamics of point particles and systems:

$$\underline{V} = V_r(t)\,\underline{e}_r + V_\theta(t)\,\underline{e}_\theta \quad -(9)$$

and Eq. (7) reduces to:

$$\frac{D}{Dt}\begin{bmatrix} V_r \\ V_\theta \end{bmatrix} = \frac{\partial}{\partial t}\begin{bmatrix} V_r \\ V_\theta \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix} \begin{bmatrix} V_r \\ V_\theta \end{bmatrix} \quad -(10)$$

3) In calculating the acceleration:

$$\underline{a} = \frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \underline{\nabla}) \underline{v} \quad -(11)$$

Here:

$$\underline{V} = \underline{v} \quad -(12)$$

and eq. (7) reduces to:

$$\frac{D}{Dt}\begin{bmatrix} v_r \\ v_\theta \end{bmatrix} = \frac{\partial}{\partial t}\begin{bmatrix} v_r \\ v_\theta \end{bmatrix} + \left(\begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r}\frac{\partial v_r}{\partial \theta} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix} \right) \begin{bmatrix} v_r \\ v_\theta \end{bmatrix} \quad -(13)$$

i.e. to the generally covariant Cartan derivative:

$$\frac{D v^a}{Dt} = \frac{\partial v^a}{\partial t} + \omega^a{}_b v^b \quad -(14)$$

where:

$$\omega^a{}_b = \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r}\frac{\partial v_r}{\partial \theta} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix} \quad -(15)$$

is the Cartan spin connection for fluid dynamics. Here

$$v^a = \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} v_r \\ v_\theta \end{bmatrix} \quad -(16)$$

is a two component column vector.

Eq. (14) is the foundation for the Navier Stokes equation, an example of Cartan geometry.

4) The general equation (7) is a well defined combination of two Cartan derivatives:

$$\frac{D\bar{V}^a}{Dt} = \frac{\partial \bar{V}^a}{\partial t} + \Omega^a{}_{b} V^b + \omega^a{}_{b} \bar{V}^b \quad -(17)$$

where:

$$\Omega^a{}_{b} = \begin{bmatrix} \frac{\partial V_r}{\partial r} & \frac{1}{r}\frac{\partial V_r}{\partial \theta} \\ \frac{\partial V_\theta}{\partial r} & \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} \end{bmatrix} \quad -(18)$$

and

$$\omega^a{}_{b} = \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix} \quad -(19)$$

By Cartan geometry:

$$\frac{Dv^a}{Dt} = \frac{\partial v^a}{\partial t} + \Omega^a{}_{b} v^b \quad -(20)$$

so

$$\boxed{\frac{D\bar{V}^a}{Dt} = \frac{\partial \bar{V}^a}{\partial t} + \left(\frac{D}{Dt} - \frac{\partial}{\partial t}\right) v^a + \omega^a{}_{b} \bar{V}^b \quad -(21)}$$

is the required generalization of particle dynamics to fluid dynamics.

For example, the Coriolis velocity of an orbit is generalized to:

$$\frac{D}{Dt}\begin{bmatrix} r \\ 0 \end{bmatrix} = \frac{\partial}{\partial t}\begin{bmatrix} r \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix}\begin{bmatrix} r \\ 0 \end{bmatrix} + \Omega^a{}_{ob}\begin{bmatrix} \dot{r} \\ r\dot{\theta} \end{bmatrix} \quad -(22)$$

and the Coriolis accelerations are generalized to:

$$\frac{D}{Dt}\begin{bmatrix} \dot{r} \\ r\dot{\theta} \end{bmatrix} = \frac{\partial}{\partial t}\begin{bmatrix} \dot{r} \\ r\dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix}\begin{bmatrix} \dot{r} \\ r\dot{\theta} \end{bmatrix} + \Omega^a{}_{ob}\begin{bmatrix} \dot{r} \\ r\dot{\theta} \end{bmatrix} \quad -(23)$$

where:

$$\underline{r}(t) \rightarrow \underline{r}(t, \theta(t), r(t)) \quad -(24)$$

and

$$\underline{v}(t) \rightarrow \underline{v}(t, r(t), \theta(t)) \quad -(25)$$

In eqs. (22) and (23):

$$\Omega^a{}_{ob} = \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r}\frac{\partial v_r}{\partial \theta} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} \end{bmatrix} := \begin{bmatrix} \Omega^1{}_{01} & \Omega^1{}_{02} \\ \Omega^2{}_{01} & \Omega^2{}_{02} \end{bmatrix} \quad -(26)$$

In the usual orbital theory:

$$\Omega^a{}_{ob} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad -(27)$$

therefore the usual result:

$$v_r = \dot{r} \quad -(28)$$

$$v_\theta = r\dot{\theta} = \omega r \quad -(29)$$

is generalized by using:

$$\frac{D}{Dt}\begin{bmatrix} r \\ 0 \end{bmatrix} = \frac{\partial}{\partial t}\begin{bmatrix} r \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix}\begin{bmatrix} r \\ 0 \end{bmatrix} + \begin{bmatrix} \Omega^1{}_{01} & \Omega^1{}_{02} \\ \Omega^2{}_{01} & \Omega^2{}_{02} \end{bmatrix}\begin{bmatrix} \dot{r} \\ r\dot{\theta} \end{bmatrix} - ($$

so

$$V_r = \dot{r} + \Omega^1{}_{01}\dot{r} + \Omega^1{}_{02}r\dot{\theta} \quad -(31)$$

$$V_\theta = \omega r + \Omega^2{}_{01}\dot{r} + \Omega^2{}_{02}r\dot{\theta} \quad -(32)$$

i.e

$$V_r = \dot{r} + \Omega^1{}_{01}\dot{r} + \Omega^1{}_{02}\,\omega r \quad -(33)$$

$$V_\theta = \omega r + \Omega^2{}_{01}\dot{r} + \Omega^2{}_{02}\,\omega r \quad -(34)$$

so

$$V_r = \left(1 + \Omega^1{}_{01}\right)\dot{r} + \Omega^1{}_{02}\,\omega r \quad -(35)$$

and

$$V_\theta = \left(1 + \Omega^2{}_{01}\right)\omega r + \Omega^2{}_{01}\dot{r} \quad -(36)$$

The complete Coriolis velocity is:

$$\underline{V} = V_r\,\underline{e}_r + V_\theta\,\underline{e}_\theta \quad -(37)$$

Therefore the generalization (24) and (25) has an effect on the orbital velocity, and on

$$V^2 = V_r^2 + V_\theta^2 \quad -(38)$$

and changes it from the Newtonian:

$$V^2 = MG\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(39)$$