362(3): Clarification of Note 362(1)

The spin connection in Note 362(1) is

$$\omega^a{}_{ob} = \begin{bmatrix} 0 & 0 \\ \dot\theta & \frac{\ddot\theta}{\dot\theta} \end{bmatrix} + \begin{bmatrix} 0 & -\dot\theta \\ \dot\theta & 0 \end{bmatrix} \quad -(1)$$

and this fixes a typo in eq. (6) of note 362(1).

Therefore:

$$\omega^a{}_{ob} = \begin{bmatrix} 0 & -\dot\theta \\ 2\dot\theta & \frac{\ddot\theta}{\dot\theta} \end{bmatrix} \quad -(2)$$

Now consider the Cartan covariant derivative:

$$\frac{Dv^a}{Dt} = \frac{dv^a}{dt} + \omega^a{}_{ob} v^b \quad -(3)$$

where

$$v^1 = v_r \quad -(4)$$

and

$$v^2 = v_\theta \quad -(5)$$

So eq. (3) is the matrix equation:

$$\frac{D}{Dt}\begin{bmatrix} v_r \\ v_\theta \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} v_r \\ v_\theta \end{bmatrix} + \begin{bmatrix} 0 & -\dot\theta \\ 2\dot\theta & \frac{\ddot\theta}{\dot\theta} \end{bmatrix}\begin{bmatrix} v_r \\ v_\theta \end{bmatrix} \quad -(6)$$

where

$$v_r = \dot r \quad -(7)$$

$$v_\theta = r\dot\theta \quad -(8)$$

So:

$$\frac{Dv_r}{Dt} = \frac{\partial v_r}{\partial t} - r\dot{\theta}^2 \quad -(9)$$

and

$$\frac{Dv_\theta}{Dt} = \frac{\partial v_\theta}{\partial t} + r\ddot{\theta} + 2\dot{r}\dot{\theta} \quad -(10)$$

here we have used:

$$\begin{bmatrix} 0 & -\dot{\theta} \\ 2\dot{\theta} & \frac{\ddot{\theta}}{\dot{\theta}} \end{bmatrix} \begin{bmatrix} \dot{r} \\ r\dot{\theta} \end{bmatrix} = \begin{bmatrix} -r\dot{\theta}^2 \\ 2\dot{r}\dot{\theta} + r\ddot{\theta} \end{bmatrix} \quad -(11)$$

and

$$\begin{bmatrix} 0 & -\dot{\theta} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{r} \\ r\dot{\theta} \end{bmatrix} = \begin{bmatrix} -r\dot{\theta}^2 \\ 0 \end{bmatrix} \quad -(12)$$

$$\begin{bmatrix} 0 & 0 \\ 2\dot{\theta} & \frac{\ddot{\theta}}{\dot{\theta}} \end{bmatrix} \begin{bmatrix} \dot{r} \\ r\dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ 2\dot{r}\dot{\theta} + r\ddot{\theta} \end{bmatrix} \quad -(13)$$

Eq. (12) is the centrifugal acceleration and Eq. (13) is the Coriolis sum of accelerations. They are the result of Cartan geometry.

In vector notation: $\quad -(14)$

$$\boxed{\underline{a} = \frac{D\underline{v}}{Dt} = \frac{Dv_r}{Dt}\,\underline{e}_r + \frac{Dv_\theta}{Dt}\,\underline{e}_\theta}$$

Therefore:

3)

$$\underline{a} = \frac{D\underline{v}}{Dt} = (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e}_\theta + \frac{\partial v_\theta}{\partial t}\underline{e}_\theta \quad -(15)$$

Therefore the conventionally defined acceleration is:

$$\boxed{\begin{aligned}\underline{a}_1 &= \frac{D\underline{v}}{Dt} - \frac{\partial v_\theta}{\partial t}\underline{e}_\theta \quad -(16)\\ &= (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e}_\theta\end{aligned}}$$

The usual acceleration is defined in plane polar coordinates by:

$$\begin{aligned}\underline{a} &= \frac{d}{dt}(v_r\underline{e}_r + v_\theta\underline{e}_\theta) \quad -(17)\\ &= \frac{\partial v_r}{\partial t}\underline{e}_r + \frac{\partial v_\theta}{\partial t}\underline{e}_\theta + v_r\frac{\partial \underline{e}_r}{\partial t} + v_\theta\frac{\partial \underline{e}_\theta}{\partial t}\\ &= \frac{\partial v_r}{\partial t}\underline{e}_r + \frac{\partial v_\theta}{\partial t}\underline{e}_\theta + v_r\dot{\theta}\underline{e}_\theta - v_\theta\dot{\theta}\underline{e}_r\\ &= \left(\frac{\partial v_r}{\partial t} - r\dot{\theta}^2\right)\underline{e}_r + \left(\frac{\partial v_\theta}{\partial t} + \dot{r}\dot{\theta}\right)\underline{e}_\theta\end{aligned}$$

4) This can be represented by:

$$\frac{D}{Dt}\begin{bmatrix} v_r \\ v_\theta \end{bmatrix} = \frac{\partial}{\partial t}\begin{bmatrix} v_r \\ v_\theta \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix}\begin{bmatrix} v_r \\ v_\theta \end{bmatrix} \quad -(18)$$

$$= \frac{\partial}{\partial t}\begin{bmatrix} \dot{r} \\ r\dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix}\begin{bmatrix} \dot{r} \\ r\dot{\theta} \end{bmatrix}$$

This is the Cartan covariant derivative:

$$\frac{D}{Dt} v^a = \frac{\partial}{\partial t} v^a + \omega^a{}_b v^b \quad -(19)$$

with

$$\omega^a{}_b = \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix} \quad -(20)$$

This spin connection is the rotation generator of the axes of the plane polar system at constant angular velocity:

$$\omega = \dot{\theta}. \quad -(21)$$

From eq. (18) it is seen that:

$$\frac{Dv_r}{Dt} = \frac{\partial v_r}{\partial t} - r\dot{\theta}^2 \quad -(21)$$

and

$$\frac{Dv_\theta}{Dt} = \frac{\partial v_\theta}{\partial t} + \dot{\theta}\dot{r} \quad -(22)$$

=) where:

$$\frac{\partial V_\theta}{\partial t} = \frac{\partial}{\partial t}(r\dot{\theta}) = \dot{r}\dot{\theta} + r\ddot{\theta} \quad - (23)$$

So:

$$\frac{DV_\theta}{Dt} = 2\dot{r}\dot{\theta} + r\ddot{\theta} \quad - (24)$$

In vector notation:

$$\underline{a} = \frac{D\underline{v}}{Dt} = \frac{DV_r}{Dt}\,\underline{e}_r + \frac{DV_\theta}{Dt}\,\underline{e}_\theta \quad - (25)$$

$$= (\ddot{r} - r\dot{\theta}^2)\,\underline{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\underline{e}_\theta$$

which is the usual expression for acceleration in plane polar coordinates, Q.E.D.

Here we have used:

$$\ddot{r} = \frac{\partial V_r}{\partial t} \quad - (26)$$

Comparing Eqs. (1) and (20) it is seen that the convective derivative leads to the additional spin connection:

$$\omega^a{}_b\,(\text{convective}) = \begin{bmatrix} 0 & 0 \\ \dot{\theta} & \frac{\ddot{\theta}}{\dot{\theta}} \end{bmatrix} \quad - (27)$$