362(4): The velocity as a Cartan covariant derivative

The velocity in plane polar coordinates is defined as the generally covariant Cartan derivative:

$$v^a = \frac{Dr^a}{Dt} = \frac{dr^a}{dt} + \omega^a{}_{b} r^b \quad -(1)$$

where

$$r^a = \begin{bmatrix} r \\ 0 \end{bmatrix} \quad -(2)$$

and

$$\omega^a{}_b = \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix} \quad -(3)$$

is the spin connection due to the rotation of the axes of the plane polar system. Similarly, the acceleration is the Cartan covariant derivative:

$$a^a = \frac{Dv^a}{Dt} = \frac{dv^a}{dt} + \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix} v^b \quad -(4)$$

where

$$v^a = \begin{bmatrix} \dot{r} \\ r\dot{\theta} \end{bmatrix} \quad -(5)$$

In general, for any vector V^a:

$$\frac{DV^a}{Dt} = \frac{dV^a}{dt} + \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix} V^b \quad -(6)$$

for the plane polar system. Note carefully that the spin connection (3) is the same for any vector.

In classical dynamics:

2)

$$v^a = v^a(t), \quad r^a = r^a(t), \quad v^a = v^a(t). \quad -(7)$$

From eqs. (1) to (3):

$$v_r = \dot{r} = \frac{dr}{dt} \quad -(8)$$

and

$$v_\theta = r\dot{\theta} = r\frac{d\theta}{dt} = \omega r \quad -(9)$$

Eq. (9) is the well known expression for the orbital velocity of a circular orbit. However, it is no longer true for an elliptical orbit.

From eqs. (8) and (9):

$$\begin{aligned}\underline{v} &= v_r \underline{e}_r + v_\theta \underline{e}_\theta \quad -(10)\\ &= \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta \\ &= \dot{r}\,\underline{e}_r + r\,\dot{\underline{e}}_r \\ &= \frac{d}{dt}\left(r\,\underline{e}_r\right)\end{aligned}$$

where we have used:

$$\dot{\underline{e}}_r = \dot{\theta}\,\underline{e}_\theta \quad -(11)$$

Using the result:

$$\underline{r} = r\,\underline{e}_r \quad -(12)$$

then

$$\underline{v} = \frac{d\underline{r}}{dt} \quad -(13)$$

3) In notes 362(2b) it was shown that eq. (12) is also true for elliptical polar coordinates in which:

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(14)$$

The use of Maxima by Dr. Horst Eckardt showed that the unit vectors $\underline{e}_r$ and $\underline{e}_\theta$ of the elliptical polar coordinate system are in general very complicated, but a great simplification of the algebra results by computer analysis, giving Eq. (12).

This result is new to vector analysis. The Coriolis velocity (1835):

$$\underline{v} = \dot{r}\,\underline{e}_r + \omega r\,\underline{e}_\theta \quad -(15)$$

is therefore due to ECE2, a generally covariant unified field theory. The Coriolis acceleration:

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{d}{dt}\left(\dot{r}\,\underline{e}_r + \omega r\,\underline{e}_\theta\right)$$

$$= \left(\ddot{r} - \omega^2 r\right)\underline{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underline{e}_\theta$$

$$-(16)$$

is also due to the generally covariant derivative of Cartan. It is very important to note that the orbital velocity, the centrifugal acceleration

4) and the Coriolis accelerations, are due to the spin connection (3) of the plane polar coordinates.

These quantities do not appear in Cartesian coordinates, and therefore depend on the motion of the coordinates. It is logical to expect that the use of the elliptical polar system will produce new physical accelerations.

In fluid dynamics, the general vector field is:

$$\underline{V} = \underline{V}(t, r, \theta) \quad -(17)$$

Its time derivative is the Lagrange derivative:

in plane polar coordinates.

$$\frac{D\underline{V}}{Dt} = \frac{\partial \underline{V}}{\partial t} + (\underline{v} \cdot \underline{\nabla})\underline{V} \quad -(18)$$

$$= \frac{\partial \underline{V}}{\partial t} + \left(v_r \frac{\partial}{\partial r} + \frac{v_\theta}{r} \frac{\partial}{\partial \theta} \right) \left(V_r \underline{e}_r + V_\theta \underline{e}_\theta \right) \quad -(19)$$

as in Note 361(5), where it was shown that:

$$\frac{D}{Dt}\begin{bmatrix} v_r \\ v_\theta \end{bmatrix} = \frac{\partial}{\partial t}\begin{bmatrix} v_r \\ v_\theta \end{bmatrix} + \left(\begin{bmatrix} \partial v_r / \partial r & \frac{1}{r}\frac{\partial v_r}{\partial \theta} \\ \partial v_\theta / \partial r & \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} \end{bmatrix} + \begin{bmatrix} 0 & -\frac{v_\theta}{r} \\ \frac{v_\theta}{r} & 0 \end{bmatrix} \right) \begin{bmatrix} v_r \\ v_\theta \end{bmatrix}$$

In the next note, Eq. (18) will be expressed in the format of Eq. (19).