62(2): The Elliptical Polar Coordinates, Position Velocity and Acceleration.

This system is defined by the ellipse:

$$X = c + r\cos\theta \quad -(1)$$

$$Y = r\sin\theta \quad -(2)$$

$$\frac{(c + r\cos\theta)^2}{a^2} + r^2\frac{\sin^2\theta}{b^2} = 1 \quad -(3)$$

with eccentricity:

$$\epsilon = \left(1 - \frac{b^2}{a^2}\right)^{1/2} \quad -(4)$$

As in previous UFT papers eqns. (1) to (4) imply:

$$r = \frac{\alpha}{1 + \epsilon\cos\theta} \quad -(5)$$

where

$$\alpha = a(1 - \epsilon^2) \quad -(6)$$

Here a and b are the semi major and minor axes and α is the half right latitude. In the Newtonian theory eq. (5) describes an object m orbiting around an object M at one focus of the ellipse. The plane polar coordinates correspond to:

$$\epsilon = 0 \quad -(7)$$

and

$$\alpha = a = b \quad -(8)$$

$$c = 0 \quad -(9)$$

with

2) It is well known that the plane polar coordinates introduce the Coriolis velocity and accelerations. These do not exist in the Cartesian coordinates. In the plane polar system:

$$\underline{r} = r \, \underline{e}_r \quad -(10)$$

$$\underline{v} = \dot{r} \, \underline{e}_r + r \dot{\theta} \, \underline{e}_\theta \quad -(11)$$

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2) \, \underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta}) \, \underline{e}_\theta \quad -(12)$$

and

$$\frac{\partial r}{\partial \theta} = 0 \quad -(13)$$

However, from eq. (5):

$$\frac{\partial r}{\partial \theta} = \frac{\epsilon r^2}{\alpha} \sin\theta \neq 0 \quad -(14)$$

so there is a self contradiction in the use of the plane polar coordinates to describe an elliptical orbit because the former describe a circular orbit.

The use of an elliptical polar coordinate system should resolve this contradiction and allow for the introduction of new forces in astronomy.

The position vector of the elliptical polar coordinate system is:

$$\underline{r} = \left(\frac{\alpha}{1 + \epsilon \cos\theta} \right) (\underline{i} \cos\theta + \underline{j} \sin\theta) \quad -(15)$$

In the usual assumption of dynamics:

$$\theta = \theta(t), \quad r = r(\theta(t)) \quad -(16)$$

Therefore the orbital velocity is:

$$\underline{v} = \left(\frac{d}{dt}\left(\frac{\alpha\cos\theta(t)}{1+\epsilon\cos\theta(t)}\right)\right)\underline{i} + \left(\frac{d}{dt}\left(\frac{\alpha\sin\theta(t)}{1+\epsilon\cos\theta(t)}\right)\underline{j}\right) \quad -(17)$$

and the orbital acceleration is:

$$\underline{a} = \left(\frac{d^2}{dt^2}\left(\frac{\alpha\cos\theta(t)}{1+\epsilon\cos\theta(t)}\right)\right)\underline{i} + \left(\frac{d^2}{dt^2}\left(\frac{\alpha\sin\theta(t)}{1+\epsilon\cos\theta(t)}\right)\right)\underline{j} \quad -(18)$$

where

$$\cos\theta(t) = \frac{1}{\epsilon}\left(\frac{\alpha}{r} - 1\right) \quad -(19)$$

and

$$\sin\theta(t) = \left(1 - \cos^2\theta(t)\right)^{1/2} \quad -(20)$$

Computer algebra can be used to work out v^2. The result in the plane polar system is:

$$v^2 = MG\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(21)$$

where:

$$\alpha = \frac{L^2}{m^2 MG} \quad -(22)$$

The unit vectors of the elliptical polar system are:

$$\underline{e}_r = \frac{\partial \underline{r}}{\partial r} \Big/ \left| \frac{\partial \underline{r}}{\partial r} \right| \quad -(23)$$

here

$$\underline{r} = r(\theta)\left(\underline{i}\cos\theta + \underline{j}\sin\theta\right) \quad -(24)$$

with

$$r(\theta) = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(25)$$

Therefore:

$$\underline{e}_r = \underline{i}\cos\theta + \underline{j}\sin\theta. \quad -(26)$$

This is the same as for the plane polar system. It follows for the elliptical polar system that:

$$\frac{\partial \underline{e}_r}{\partial r} = \underline{0} \quad -(27)$$

as for the plane polar system.

However,

$$\frac{\partial \underline{r}}{\partial \theta} = \underline{i}\left(\frac{\partial r}{\partial \theta}\cos\theta - r\sin\theta\right) + \underline{j}\left(\frac{\partial r}{\partial \theta}\sin\theta + r\cos\theta\right) \quad -(28)$$

Therefore the contradiction between eqs (13) and (14) is resolved because $\partial r / \partial \theta$ is not zero and is given by eq. (5)

5) From eq. (28):

$$\left|\frac{\partial r}{\partial\theta}\right| = \left(\left(\frac{\partial r}{\partial\theta}\cos\theta - r\sin\theta\right)^2 + \left(\frac{\partial r}{\partial\theta}\sin\theta + r\cos\theta\right)^2\right)^{1/2}$$

$$= \left(r^2 + \left(\frac{\partial r}{\partial\theta}\right)^2\right)^{1/2} \quad -(29)$$

The unit vector $\underline{e}_\theta$ of the elliptical polar system is therefore:

$$\underline{e}_\theta = \frac{\left[\underline{i}\left(\frac{\partial r}{\partial\theta}\cos\theta - r\sin\theta\right) + \underline{j}\left(\frac{\partial r}{\partial\theta}\sin\theta + r\cos\theta\right)\right]}{\left(r^2 + \left(\frac{\partial r}{\partial\theta}\right)^2\right)^{1/2}} \quad -(30)$$

$$:= A(\theta)\,\underline{i} + B(\theta)\,\underline{j} \quad -(31)$$

Here:

$$A(\theta) = \left(\frac{\partial r}{\partial\theta}\cos\theta - r\sin\theta\right)\left(r^2 + \left(\frac{\partial r}{\partial\theta}\right)^2\right)^{-1/2} \quad -(32)$$

$$B(\theta) = \left(\frac{\partial r}{\partial\theta}\sin\theta + r\cos\theta\right)\left(r^2 + \left(\frac{\partial r}{\partial\theta}\right)^2\right)^{-1/2} \quad -(33)$$

and

$$\frac{\partial r}{\partial\theta} = \frac{\epsilon r^2}{\alpha}\sin\theta \quad -(34)$$

These results are consistent with orbital theory.

6) From eqs. (26) and (31)...

$$\underline{e}_r = \underline{i}\cos\theta + \underline{j}\sin\theta \quad -(35)$$

$$\underline{e}_\theta = A(\theta)\,\underline{i} + B(\theta)\,\underline{j} \quad -(36)$$

It follows that:

$$\underline{j} = \left(\frac{\underline{e}_r}{\cos\theta} - \frac{\underline{e}_\theta}{A(\theta)}\right)\left(\frac{\sin\theta}{\cos\theta} - \frac{B(\theta)}{A(\theta)}\right)^{-1} \quad -(37)$$

$$\underline{i} = \left(\frac{\underline{e}_r}{\sin\theta} - \frac{\underline{e}_\theta}{B(\theta)}\right)\left(\frac{\cos\theta}{\sin\theta} - \frac{A(\theta)}{B(\theta)}\right)^{-1} \quad -(38)$$

Therefore:

$$\underline{i} = C(\theta)\,\underline{e}_r - D(\theta)\,\underline{e}_\theta \quad -(39)$$

$$\underline{j} = E(\theta)\,\underline{e}_r - F(\theta)\,\underline{e}_\theta \quad -(40)$$

where:

$$C(\theta) = \frac{1}{\sin\theta}\left(\frac{\cos\theta}{\sin\theta} - \frac{A(\theta)}{B(\theta)}\right)^{-1} \quad -(41)$$

$$D(\theta) = \frac{1}{B(\theta)}\left(\frac{\cos\theta}{\sin\theta} - \frac{A(\theta)}{B(\theta)}\right)^{-1} \quad -(42)$$

$$E(\theta) = \frac{1}{\cos\theta}\left(\frac{\sin\theta}{\cos\theta} - \frac{B(\theta)}{A(\theta)}\right)^{-1} \quad -(43)$$

$$F(\theta) = \frac{1}{A(\theta)}\left(\frac{\sin\theta}{\cos\theta} - \frac{B(\theta)}{A(\theta)}\right)^{-1} \quad -(44)$$

7) It follows that:

$$\underline{r} = r(\theta)\left(\underline{i}\cos\theta + \underline{j}\sin\theta\right) \quad -(45)$$

$$= \left(\frac{\alpha}{1+\epsilon\cos\theta}\right)\Big(\left(C(\theta)\underline{e}_r - D(\theta)\underline{e}_\theta\right)\cos\theta \quad -(46)$$

$$+ \left(E(\theta)\underline{e}_r - F(\theta)\underline{e}_\theta\right)\sin\theta\Big)$$

$$\boxed{\underline{r} = f_1\,\underline{e}_r \mp f_2\,\underline{e}_\theta} \quad -(47)$$

where:

$$f_1 = \left(\frac{\alpha}{1+\epsilon\cos\theta}\right)\left(C(\theta)\cos\theta + E(\theta)\sin\theta\right) \quad -(48)$$

$$f_2 = \left(\frac{\alpha}{1+\epsilon\cos\theta}\right)\left(D(\theta)\cos\theta + F(\theta)\sin\theta\right) \quad -(49)$$

These can be transformed into functions of r using eqs. (19) and (20), so:

$$\underline{r} = f_1(r)\,\underline{e}_r - f_2(r)\,\underline{e}_\theta \quad -(50)$$

and

$$\boxed{\underline{r} \neq r\,\underline{e}_r} \quad -(51)$$

in the elliptical polar system.

). It follows that:

$$\underline{v} = \frac{d\underline{r}}{dt} = \frac{df_1(r)}{dt}\underline{e}_r + f_1(r)\frac{d\underline{e}_r}{dt} - \frac{df_2(r)}{dt}\underline{e}_\theta - f_2(r)\frac{d\underline{e}_\theta}{dt} \quad -(52)$$

and:

$$\underline{a} = \frac{d\underline{v}}{dt} \quad -(53)$$

Therefore the elliptical polar system of coordinates does indeed produce new velocities and accelerations, not inferred by Coriolis who used a plane polar system.

Example: Circular Orbit

In this case eq. (17) is:

$$\underline{v} = \alpha\left(\left(\frac{d}{dt}\cos\theta(t)\right)\underline{i} + \left(\frac{d}{dt}\sin\theta(t)\right)\underline{j}\right) \quad -(54)$$

so

$$v^2 = \alpha^2\dot{\theta}^2 \quad -(55)$$

From a Lagrangian analysis:

$$\dot{\theta} = \omega = L/(mr^2) \quad -(56)$$

and $\alpha = r$, so

$$\boxed{v = \omega r} \quad -(57)$$

and

$$v^2 = \frac{MG}{r} = \frac{L^2}{m^2r^2} = \omega^2r^2 \quad -(58)$$

QED.