362(1): Reduction of the Convective Derivative to the Plane Polar System

If the convective derivative is used, the acceleration is:

$$\underline{a} = \frac{D\underline{v}}{Dt} = \frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v} \quad -(1)$$

where

$$\underline{v} = \underline{v}(t, r(t), \theta(t)). \quad -(2)$$

If the plane polar system is used:

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\,\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\underline{e}_\theta \quad -(3)$$

where

$$\underline{v} = \underline{v}(t).$$

Eqs. (1) and (3) are Cartan covariant derivatives:

$$\frac{Dv^a}{Dt} = \frac{\partial v^a}{\partial t} + \omega^a{}_{0b} v^b \quad -(4)$$

For eq. (1):

$$\omega^a{}_{0b} = \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r}\frac{\partial v_r}{\partial \theta} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix} \quad -(5)$$

and for eq. (3):

$$\omega^a{}_{0b} = \begin{bmatrix} 0 & 0 \\ \dot{\theta} & \frac{r\ddot{\theta}}{\dot{\theta}} \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix} \quad -(6)$$

2)

Here:

$$V^1 = V_r = \dot{r} \quad -(7)$$

$$V^2 = V_\theta = r\dot{\theta} \quad -(8)$$

For eq. (1) and eq. (2):

$$\frac{D}{Dt}\begin{bmatrix} V_r \\ V_\theta \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} V_r \\ V_\theta \end{bmatrix} + \omega^a{}_b \begin{bmatrix} V_r \\ V_\theta \end{bmatrix} \quad -(9)$$

Eq. (1) produces the accelerations:

$$\underline{a}_1 = \left(\dot{r}\frac{d\dot{r}}{dr} + \dot{\theta}\frac{d\dot{r}}{d\theta} \right)\underline{e}_r + \left(r\dot{r}\frac{d\dot{\theta}}{dr} + \dot{\theta}^2\frac{dr}{d\theta} \right)\underline{e}_\theta \quad -(10)$$

but from eq. (2):

$$\underline{a}_1 = \underline{0}. \quad -(11)$$

The result (11) is the consequence of the following property of the plane polar coordinate system:

$$\frac{dr}{d\theta} = 0. \quad -(12)$$

It follows that r and θ are independent, i.e. the coordinate system (r, θ) is defined in such a way that r is not a function of θ. The proof of eq. (12) follows from:

$$X = r\cos\theta \quad -(13)$$

$$Y = r\sin\theta \quad -(14)$$

so:

$$\underline{r} = X\underline{i} + Y\underline{j} = \underline{i}\, r\cos\theta + \underline{j}\, r\sin\theta \quad -(15)$$

So:

$$\frac{\partial \underline{r}}{\partial r} = r\left(\underline{i}\cos\theta + \underline{j}\sin\theta\right) \quad -(16)$$

$$\frac{\partial \underline{r}}{\partial \theta} = r\left(-\underline{i}\sin\theta + \underline{j}\cos\theta\right) \quad -(17)$$

The unit vectors of the plane polar system are:

$$\underline{e}_r = \frac{\partial \underline{r}}{\partial r} \Big/ \left|\frac{\partial \underline{r}}{\partial r}\right| = \underline{i}\cos\theta + \underline{j}\sin\theta \quad -(18)$$

$$\underline{e}_\theta = \frac{\partial \underline{r}}{\partial \theta} \Big/ \left|\frac{\partial \underline{r}}{\partial \theta}\right| = -\underline{i}\sin\theta + \underline{j}\cos\theta \quad -(19)$$

It follows that:

$$\frac{\partial \underline{e}_r}{\partial \theta} = \underline{e}_\theta, \quad \frac{\partial \underline{e}_\theta}{\partial \theta} = -\underline{e}_r \quad -(20).$$

Now note that:

$$\underline{e}_r = \frac{\partial \underline{r}}{\partial r} = \frac{\partial}{\partial r}\left(r\underline{e}_r\right) = \underline{e}_r + r\frac{\partial \underline{e}_r}{\partial r} \quad -(21)$$

So

$$\frac{\partial \underline{e}_r}{\partial r} = \underline{0} \quad -(22)$$

Secondly:

$$\underline{e}_\theta = \frac{1}{r}\frac{\partial \underline{r}}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial \theta}\left(r\underline{e}_r\right)$$

$$= \frac{1}{r}\frac{\partial r}{\partial \theta}\underline{e}_r + \frac{\partial \underline{e}_r}{\partial \theta} \quad -(23)$$

$$= \frac{1}{r}\frac{\partial r}{\partial \theta}\underline{e}_r + \underline{e}_\theta$$

So

$$\frac{\partial r}{\partial \theta} = 0 \quad -(24)$$

Q.E.D.

4) Therefore the plane polar system is defined by

$$\left. \begin{array}{l} X = r\cos\theta,\ Y = r\sin\theta,\ X^2 + Y^2 = r^2, \\ \partial \underline{e}_r / \partial r = 0,\ \partial \dot{r} / \partial\theta = 0 \end{array} \right\} - (25)$$

and by eqns. (20). In dynamics:

$$\underline{r} = \underline{r}(t) \quad - (26)$$

and

$$\underline{v}(t) = \frac{d\underline{r}(t)}{dt} = \frac{d}{dt}(r\underline{e}_r) \quad - (27)$$

$$= \frac{dr}{dt}\underline{e}_r + r\frac{d\theta}{dt}\underline{e}_\theta$$

It follows that in the plane polar system:

$$\frac{\partial\theta}{\partial r} = 0 \quad - (28)$$

Therefore:

$$\frac{\partial\dot{\theta}}{\partial r} = \frac{\partial\theta}{\partial r}\frac{\partial\dot{\theta}}{\partial\theta} = 0, \quad - (29)$$

$$\frac{\partial\dot{r}}{\partial r} = \frac{\partial\dot{r}}{\partial\theta}\frac{\partial\theta}{\partial r} = 0 \quad - (30)$$

and

$$\frac{\partial\dot{r}}{\partial\theta} = \frac{\partial r}{\partial\theta}\frac{\partial\dot{r}}{\partial r} = 0 \quad - (31)$$

and

$$\underline{a}_1 = \underline{0} \quad - (32)$$

Q.E.D.

Therefore the use of the covariant derivative defined by eqs. (1) and (2) introduces a more general coordinate system.

This realization allows the development of orbital theory in terms of the coordinate system itself.

It can be seen that the property (24) of the plane polar coordinates corresponds to a circular orbit, defined by

$$r = \frac{\alpha}{1 + \epsilon \cos\theta} \quad -(33)$$

with

$$\epsilon = 0 \quad -(34)$$

so

$$r = \alpha = \text{constant} \quad -(35)$$

and

$$dr / d\theta = 0 \quad -(36)$$

Q.E.D.

In general however:

$$\epsilon \neq 0 \quad -(37)$$

so

$$\frac{dr}{d\theta} = \frac{\epsilon r^2}{\alpha} \sin\theta \quad -(38)$$

The result (38) is the classical way of linking of orbit as a function defined in a coordinate system. The function has the property (38), the coordinate system has the property (24). They are the same if and only if the function is the circle, defined by eqs. (13) and (14). The next note will consider the orbit given by eq. (1) in general.